HYPER VERSION

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The O(dd) story of massive supergravity

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ABSTRACT: The low energy effective action describing the standard Kaluza-Klein reduction of heterotic string theory on a d-torus possesses a manifest $\mathrm{O}(d, d + 16)$ symmetry. We consider generalized Scherk-Schwarz reductions of the heterotic string to construct massive gauged supergravities. We show that the resulting action can still be written in a manifestly $\mathrm{O}(d, d + 16)$ invariant form, however, the U-duality transformations also act on the mass parameters. The latter play the dual role of defining the scalar potential and the nonabelian structure constants. We conjecture that just as for the standard reduction, a subgroup of this symmetry corresponds to an exact duality symmetry of the heterotic string theory.

KEYWORDS: Superstrings and Heterotic Strings, String Duality, Superstring Vacua, Supergravity Models.

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1. Introduction

Recently progress in understanding string dualities and the role of p-branes has lead to interest in constructing massive supergravity theories through "unconventional" compactifications of massless supergravities in higher dimensions [1, 2, 3, 4]. The seminal insight made in ref. [1] was realizing that given a theory containing an axion, that is a massless scalar with only derivative couplings, a consistent compactification could be made in which the axion was given a linear dependence on the internal coordinates. While the reduced action remains independent of the internal coordinates, the slope parameters of this linear dependence appear as mass parameters in the reduced theory. As realized in [1], these reductions are actually a special class within the general framework developed by Scherk and Schwarz[5] for producing masses from dimensional reduction. The Scherk-Schwarz approach focuses on the global symmetries, i.e. the U-duality symmetries, of the action, and the dependence on the internal coordinates takes the form of a U-duality transformation that varies (in a specific way) over the internal space. For the generalized axion reductions above, the relevant symmetry is the shift symmetry of the scalar axion, and the fact that the corresponding symmetry generator is nilpotent yields the simple linear dependence mentioned above [6]. Various applications and extensions of the generalized reductions with axionic masses were explored in [2, 3, 4, 6, 7, 8, 9].

However, the picture arising from the exploratory investigations of [1] was one of many disjoint massive supergravities in lower dimensions. This situation contrasts with the prevailing theme in string theory of recent years in which U-duality has played a central role in unifying disparate (supergravity and superstring) theories as various phases of a single U-theory[12, 13]. In part, the fragmented picture for the massive supergravities appeared because the linear ansatz described above could only accommodate specific combinations of masses. That is in many instances where the higher dimensional theory contained a number of axions, all of the corresponding mass parameters could not be simultaneously turned on within this scheme. This point was clarified in [6], where it was shown that the problem arose because the corresponding axionic symmetries did not commute. However, any combination of masses was easily accommodated within the Scherk-Schwarz framework, although the reduction ansatz now involved a polynomial (or even more general) dependence on the internal coordinates. Given that there is no restriction on the types of axionic masses, it was conjectured [6] that various distinct massive supergravities should all be a part of a single U-duality invariant massive theory.

In the present paper, we demonstrate how the preceding conjecture is realized for generalized toroidal compactifications of heterotic string theory. In this case, the standard Kaluza-Klein reduction on a d-dimensional torus from 10 to $10-d$ dimensions produces a theory with global $\mathrm{O}(d, d{+}16)$ symmetry and with a $\mathrm{U}(1)^{2d+16}$ gauge group. As shown by ref. [24], the effective action can be organized to make the former U-duality symmetry manifest. Essentially the $2d+16$ gauge fields may be assembled as a vector under this symmetry, while there are $d(d+16)$ moduli scalars transforming as a traceless symmetric tensor. We will show that this global symmetry is retained in the massive theories produced by generalized Scherk-Schwarz reductions. The bosonic part of effective action may be written as:

$$S = \int d^D x\sqrt{-g}e^{-\phi}\Big\{R + (\nabla\phi)^2 + \frac{1}{8}L_{ab}\mathcal{D}_\mu M^{bc}L_{cd}\mathcal{D}^\mu M^{da} - \\ - \frac{1}{4}F^a_{\mu\nu}L_{ab}M^{bc}L_{cd}F^{d\mu\nu} - \frac{1}{12}H^2_{\mu\nu\lambda} - W(M)\Big\}\,, \qquad (1.1)$$

where the scalar potential takes the simple form:

$$W(M) = \frac{1}{12}M^{ad}M^{be}M^{cf}f_{abc}f_{def} - \frac{1}{4}M^{ad}L^{be}L^{cf}f_{abc}f_{def}\,. \qquad (1.2)$$

(The reader is invited to read the main text for a full explanation of this result.) The essential point is that the various mass parameters introduced by the generalized reduction can be organized as a completely antisymmetric three-index tensor f_{abc} under the $\mathrm{O}(d, d+16)$ transformations. These parameters play a dual role in the reduced theory: first, as mass parameters defining the scalar potential (1.2), and second as structure constants in the *non-abelian* gauge group of this theory, implicitly appearing in $F^a_{\mu\nu}$ and $\mathcal{D}^\mu M^{ab}$. That is the generalized reduction has produced a gauged supergravity with a nontrivial non-abelian symmetry.

A simple intuition which explains (at least in part) the emergence of this non-abelian symmetry is as follows: the Scherk-Schwarz reduction introduces an axionic shift which depends on internal coordinates. Now a part of the gauge symmetry in the reduced theory can be thought of as local shifts of the internal coordinates. These Kaluza-Klein gauge transformations are inherited from the diffeomorphism invariance of original ten-dimensional theory. Hence consistency of this symmetry in the generalized reduction requires that these gauge transformations be accompanied by a *local* axionic shift. That is the latter symmetries, which are "ordinarily" only a part of the global U-duality group, have now been incorporated as a part of the local gauge group. Further given that these axionic symmetries in general do not commute [6], it must be that the gauge group is modified to become non-abelian.

For the standard Kaluza-Klein reduction on T^d, the $\mathrm{O}(d, d+16, \mathbb{R})$ transformations map one configuration of background space-time fields to another configuration. Now naively, it may appear that the U-duality symmetry is broken in the reduced action (1.1) by the appearance of the mass parameters f_{abc}. However, this is only spontaneous symmetry breaking. In the present case, we know that these couplings are simply associated with the presence of additional nontrivial background fields in the internal space. Hence our construction reveals that U-duality symmetry consists of the usual transformation rules for the fields in the $(10-d)$-dimensional space-time supplemented by a compensating transformation of the mass parameters, i.e. the nontrivial fields on the internal space. The action (1.1) maintains U-duality invariance, however, the transformations also act on the couplings f_{abc} in the obvious way. A novel feature in the case of the generalized reductions is then that in transforming the internal fields, U-duality maps one reduced theory to another with modified couplings. This U-duality covariant formalism provides a unified framework incorporating all of the previously "distinct" massive supergravities, which various generalized reductions could have produced for low energy heterotic string theory. It is natural to conjecture that the $\mathrm{O}(d, d+16, \mathbb{Z})$ subgroup, which is an exact symmetry of the full heterotic string theory with a standard toroidal compactification[29], will remain an exact symmetry of the full string theory for the generalized axion reductions — we mention some subtleties in section 6.

Some of our results above can be seen in the previous work of ref. [3], where a simpler Scherk-Schwarz reduction of the effective action of heterotic string theory produced a gauged $N=4$ supergravity theory in four dimensions. In the context of nine-dimensional massive Type II supergravity, ref. [9] has constructed an effective action which is manifestly invariant under the relevant U-duality group, namely SL$(2,\mathbb{R})$. Hull[10] considered the effect of duality transformations and their singular limits on gauged supergravities. There the singular limits were used to construct new theories. Further, Boucher[11] also considered dimensional reductions of $D=11$ supergravity similar to those constructed here.

The remainder of our paper is organized as follows: In section 2, we review the standard Kaluza-Klein reduction of low energy heterotic string theory on a d-torus. In particular, we describe the global SO$(d,d+16)$ symmetries, which form the U-duality group for the reduced theory. Section 3 presents a discussion of generalized reductions which include constant fluxes of the three-form or gauge field strengths on the torus. The reduced action for these compactifications is assembled in the form of eq. (1.1), in which the SO$(d,d+16)$ invariance remains manifest. Section 4 describes a generalized reduction ansatz which introduces curvatures in the internal geometry, and produces masses for the metric axions in the reduced theory. Section 5 then describes the general massive reduction involving all three sources of the masses. Again, the U-duality invariant form of the action (1.1) is recovered with a modified set of structure constants f_{abc}. Section 6 provides a discussion of our results. In particular, we note that the formalism introduced in section 4 and the resulting low energy action applies for more general internal geometries than the d-torus. This is followed by a number of appendices, which contain details of the calculations made in performing the generalized reductions. A final appendix presents a new perspective on discussion of the generalized axion reductions in ref. [6].

2. A review of Kaluza-Klein reduction

We begin with a review of the standard Kaluza-Klein reduction of low energy heterotic string theory on a d-torus. Our notation will be such that $d+D=10$ and hence this compactification yields an effective D-dimensional theory. In ten dimensions, the low energy action is

$$S=\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\Big\{\mathcal{R}+(\nabla\Phi)^2-\frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}-\frac{1}{4}\Sigma_{I=1}^{16}\mathcal{F}^I{}_{\mu\nu}\mathcal{F}^{I\,\mu\nu}\Big\}\,. \tag{2.1}$$

The ten-dimensional fields, Φ, $\mathcal{G}_{\mu\nu}$, $\mathcal{R}$, $\mathcal{H}_{\mu\nu\lambda}$ and $\mathcal{F}^I{}_{\mu\nu}$, denote the dilaton, (string-frame) metric, Ricci scalar, Kalb-Ramond three-form field strength, and the Yang-Mills field strengths, respectively. The D-dimensional counterparts of these fields will be denoted with upper case latin letters, except the dilaton, which will be ϕ. Our convention for the metric signature is $\mathcal{G}=(-,+,+,\ldots,+)$, and that for the curvature is $R^\mu{}_{\nu\lambda\sigma}=\partial_\lambda\Gamma^\mu_{\nu\sigma}-\partial_\sigma\Gamma^\mu_{\nu\lambda}+\cdots$ We assume that the only nontrivial components of the Yang-Mills potential reside in the Cartan subalgebra of the gauge group,[1] and hence ${\mathcal F^I}_{\mu\nu} = \partial_\mu {\mathcal A^I}_\nu - \partial_\nu {\mathcal A^I}_\mu$. The low energy action (2.1) has been truncated to terms with at most two derivatives. Consistent with this truncation, the three-form $\mathcal H$ is defined by including only the Yang-Mills Chern-Simons term,

$$\mathcal H = d\mathcal B - \frac{1}{2}\Sigma_{I=1}^{16}\mathcal A^I \wedge \mathcal F^I \,. \tag{2.2}$$

In component notation, we have $\mathcal H_{\mu\nu\rho} = \partial_\mu \mathcal B_{\nu\rho} - \frac{1}{2}\Sigma_{I=1}^{16}\mathcal A^I_\mu {\mathcal F^I}_{\nu\rho} +$ *cyclic permutations*, by the antisymmetry of $\mathcal B_{\mu\nu}$ and ${\mathcal F^I}_{\mu\nu}$. The normalization in eq. (2.2) corresponds to choosing $\alpha' = 1$.

We wish to consider the standard Kaluza-Klein dimensional reduction of heterotic action (2.1) on a d-torus, to set the stage for Scherk-Schwarz reductions in the following sections. The starting point for any compactification is a decomposition of the tensor degrees of freedom [14]

$$\begin{aligned}
d\mathcal S^2 &= g_{\mu\nu}(x,y)dx^\mu dx^\nu + \hat{\mathcal G}_{MN}(x,y)\left(dy^M + {\hat V^M}_\mu(x,y)dx^\mu\right)\left(dy^N + {\hat V^N}_\nu(x,y)dx^\nu\right)\\
\mathcal B &= \frac{1}{2}\hat{\mathcal B}_{\mu\nu}(x,y)dx^\mu dx^\nu + \hat{\mathcal B}_{\mu M}(x,y)dx^\mu dy^M + \frac{1}{2}\hat{\mathcal B}_{MN}(x,y)dy^M dy^N\\
\mathcal A^I &= {\hat{\mathcal A}^I}_\mu(x,y)dx^\mu + {\hat{\mathcal A}^I}_M(x,y)dy^M \,,
\end{aligned} \tag{2.3}$$

where x^μ and y^M denote the coordinates on the uncompactified and the compact internal subspaces, respectively, with $\mu = 0, 1, \ldots, D-1$ and $M = 1, 2, \ldots, d$. As usual, there is a summation when such indices appear repeated in a subscript-superscript pair. Our convention will be that the gauge group indices, $I, J = 1, 2, \ldots, 16$, always appear as superscripts, and in the following, a summation will also be implied for a pair of repeated gauge superscripts. Note that there are no assumptions about the structure of the space-time involved in writing eq. (2.3). For a Kaluza-Klein reduction on a torus, the vector fields ∂_M generate isometries of the system. Naively, this translates into the statement that none of fields depend on the y^M's. Specifically, this means that

$$\begin{array}{lll}
\hat g_{\mu\nu}(x,y) = g_{\mu\nu}(x) & \hat{\mathcal B}_{\mu\nu}(x,y) = \mathcal B_{\mu\nu}(x) & \\
{\hat V^M}_\mu(x,y) = {V^M}_\mu(x) & \hat{\mathcal B}_{\mu M}(x,y) = \mathcal B_{\mu M}(x) & {\hat{\mathcal A}^I}_\mu(x,y) = {\mathcal A^I}_\mu(x)\\
\hat{\mathcal G}_{MN}(x,y) = \mathcal G_{MN}(x) & \hat{\mathcal B}_{MN}(x,y) = \mathcal B_{MN}(x) & {\hat{\mathcal A}^I}_M(x,y) = {\mathcal A^I}_M(x)\\
\Phi(x,y) = \phi(x) + \frac{1}{2}\ln|\det(\mathcal G_{MN}(x))| & &
\end{array} \tag{2.4}$$

The dimensional reduction produces a number of new scalar and vector fields. The additional scalars are the internal components of the metric, the two-form and the gauge fields, i.e. the fields in the third line in eq. (2.4). For the compactification on a d-torus, the counting of these scalars is: $d(d+1)/2$ from the metric, $d(d-1)/2$

[1] Wilson lines in a generic toroidal compactification will break all but the corresponding Abelian gauge symmetry.

from the two-form, and $16d$ from the 16 gauge fields. Thus there are a total of $d(d+16)$ moduli scalars in the reduced theory.

The case of the vectors is more interesting. The off-diagonal terms in the metric and the two-form, $V^M{}_\mu$ and $\mathcal{B}_{\mu M}$, transform as vector fields with respect to the spacetime diffeomorphisms of the D-dimensional theory, and so give rise to new U(1) gauge fields. A closer scrutiny of eq. (2.3) shows that the split of the components of $\mathcal{B}$ and $\mathcal{A}^I$ as given there is not "gauge-invariant" as it stands. The forms dy^M transform under residual diffeomorphisms according to $dy^M \to dy'^M = dy^M + d\omega^M(x)$. This translates into the Kaluza-Klein gauge transformations $V^M{}_\mu \to V'^M{}_\mu = V^M{}_\mu - \partial_\mu \omega^M$, which ensure the invariance of the internal space d-bein, i.e. $dy^M + V^M{}_\mu dx^\mu \to dy^M + V^M{}_\mu dx^\mu$. However, in the decomposition of eq. (2.3), the reduced Yang-Mills gauge fields and the vectors $\mathcal{B}_{\mu M}$ transform nontrivially under this symmetry. To demonstrate manifest gauge invariance, one's best route is to consider the kinetic terms in the action on the tangent space, and carry dimensional reduction there, using the gauge-invariant vielbein. This is the standard procedure used for the dimensional reduction of supergravity theories [15]. When the result of this calculation is pulled back to the holonomic D-dimensional basis, it is guaranteed to be manifestly gauge invariant. After straightforward but tedious algebraic manipulations with the reduction formulas, we find that the reduced degrees of freedom, with simple gauge transformation properties, are given in terms of the original higher-dimensional degrees of freedom as follows:

$$\begin{aligned} A^I{}_\mu &= \mathcal{A}^I{}_\mu - \mathcal{A}^I{}_M V^M{}_\mu \\ B_{\mu M} &= \mathcal{B}_{\mu M} + \mathcal{B}_{MN} V^N{}_\mu + \frac{1}{2}\mathcal{A}^I{}_M A^I_\mu \\ B_{\mu\nu} &= \mathcal{B}_{\mu\nu} + V^M{}_{[\mu} B_{\nu]M} - \mathcal{B}_{MN} V^M{}_\mu V^N{}_\nu - \mathcal{A}^I{}_M V^M{}_{[\mu} A^I{}_{\nu]} \,. \end{aligned} \tag{2.5}$$

In appendix B, we give the generalization of the calculation leading to eq. (2.5) for the case when various axionic masses are turned on. Eq. (2.5) can be easily deduced from there by setting all masses to zero. Hence we will merely quote the result here. The vector fields $B_{\mu M}$ and $A^I{}_\mu$ together with $V^M{}_\mu$ comprise the full multiplet of $2d+16$ Abelian U(1) gauge fields, with simple, decoupled, gauge transformation properties. Their field strengths will be denoted: $V^M{}_{\mu\nu} = \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu$, $H_{\mu\nu M} = \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M}$ and $F^I{}_{\mu\nu} = \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu$.

The reduced action in D dimensions may be decomposed as follows [14]:

$$S = S_1 + S_2 + S_3 \,, \tag{2.6}$$

where the reduced metric-dilaton-two-form action is

$$S_1 = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 - \frac{1}{12} H^2_{\mu\nu\lambda} \Big\} \,, \tag{2.7}$$

the scalar moduli action is

$$S_2 = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ \frac{1}{4} (\nabla_\mu \mathcal{G}_{MN})(\nabla^\mu \mathcal{G}^{MN}) - \frac{1}{2} \mathcal{G}^{MN} (\nabla_\mu \mathcal{A}^I{}_M)(\nabla^\mu \mathcal{A}^I{}_N) - \\ - \frac{1}{4} \mathcal{G}^{MP} \mathcal{G}^{NQ} \Big(\nabla_\mu \mathcal{B}_{MN} + \mathcal{A}^I{}_{[M} \nabla_\mu \mathcal{A}^I{}_{N]} \Big) \Big(\nabla^\mu \mathcal{B}_{PQ} + \mathcal{A}^J{}_{[P} \nabla^\mu \mathcal{A}^J{}_{Q]} \Big) \Big\} \tag{2.8}$$

and the gauge field action is

$$S_3 = -\frac{1}{4} \int d^D x \sqrt{-g} e^{-\phi} \Big\{ f^I_{\mu\nu} f^{I\,\mu\nu} + \mathcal{G}^{MN} h_{\mu\nu M} h^{\mu\nu}{}_N + \mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\,\mu\nu} \Big\}. \tag{2.9}$$

In the latter, we use the definitions

$$f^I_{\mu\nu} = F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu}\,, \qquad h_{\mu\nu M} = H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MN} V^N{}_{\mu\nu}\,, \tag{2.10}$$

where $\mathcal{C}_{MN} = \mathcal{B}_{MN} + \frac{1}{2} \mathcal{A}^I{}_M \mathcal{A}^I{}_N$. The reduced three-form field strength in eq. (2.7) is

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} A^I{}_\mu F^I{}_{\nu\rho} - \frac{1}{2} V^M{}_\mu H_{\nu\rho M} - \frac{1}{2} B_{\mu M} V^M{}_{\nu\rho} + \text{ cyclic perm}. \tag{2.11}$$

In addition to the original Yang-Mills Chern-Simons terms, the three-form field strength now also contains the induced Chern-Simons terms arising for the new gauge fields appearing in the reduction. These additional terms are essential in establishing the duality symmetries (see below) as shown in [14], and remain important in that context when higher derivative corrections are also included [18].

The reduced low energy theory has a global $\mathrm{O}(d, d+16, \mathbb{R})$ symmetry [14, 16]. This symmetry is a generalization of T-duality symmetry, which interchanges string momentum and winding modes [17]. In order to make this duality symmetry manifest, we must introduce some additional notation. Towards this end, let us define a new set of (lower-case Latin) indices which take values $a, b = 1, 2, \ldots, 2d+16$. Then $\mathrm{O}(d, d+16)$ transformations may be defined as any real matrices $\Omega^a{}_b$ leaving invariant the $\mathrm{O}(d, d+16)$ metric

$$\Omega^a{}_c \Omega^b{}_d L^{cd} = L^{ab}\,, \qquad \text{with } L^{ab} = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1}' \end{pmatrix} \tag{2.12}$$

where $\mathbf{1}$ are $d \times d$ unit matrices and $\mathbf{1}'$ a 16×16 unit matrix. In more conventional matrix notation, we may write eq. (2.12) as $\Omega\, L\, \Omega^T = L$, where the superscript T indicates matrix transposition. We will denote the inverse of the metric (2.12) as L_{ab}, even though it is precisely the same matrix.

Now following [14, 16], we can define the reduced gauge multiplet according to

$$A^a_\mu = \begin{pmatrix} V^M{}_\mu \\ B_{\mu M} \\ A^I{}_\mu \end{pmatrix}, \qquad F^a_{\mu\nu} = \begin{pmatrix} V^M{}_{\mu\nu} \\ H_{\mu\nu M} \\ F^I{}_{\mu\nu} \end{pmatrix}. \tag{2.13}$$

In keeping with the index notation, these gauge fields transform in the fundamental representation of $\mathrm{O}(d,d+16)$, i.e. $A^a \to A'^a = \Omega^a{}_b A^b$ and $F^a \to F'^a = \Omega^a{}_b F^b$. From the notation introduced here, we also note that it is useful to keep in mind that an $\mathrm{O}(d,d+16)$ superscript runs over a contravariant index on the internal d-torus, a covariant index on the same space, and a gauge index labeling the sixteen Cartan generators, i.e. $X^a = \{X^M, X_M, X^I\}$. Lowering the $\mathrm{O}(d,d+16)$ superscript with the inverse of the invariant metric (2.12) interchanges the first two sets of components, e.g., $X_a = L_{ab}X^b = \{X_M, X^M, X^I\}$.

The scalar moduli fields, which parameterize the internal components of the metric, two-form and the Yang-Mills fields, may also be assembled in a form covariant under the duality transformations. Using the $d \times d$ matrices $\mathcal{G} = (\mathcal{G}_{MN})$, $\mathcal{B} = (\mathcal{B}_{MN})$, and mixed $d \times 16$ matrix $\mathcal{A} = (\mathcal{A}^I{}_M)$, we can introduce the matrix [14, 16]

$$M^{ab} = \begin{pmatrix} \mathcal{G}^{-1} & -\mathcal{G}^{-1}\mathcal{C} & -\mathcal{G}^{-1}\mathcal{A} \\ -\mathcal{C}^T\mathcal{G}^{-1} & \mathcal{G} + \mathrm{a} + \mathcal{C}^T\mathcal{G}^{-1}\mathcal{C} & \mathcal{A} + \mathcal{C}^T\mathcal{G}^{-1}\mathcal{A} \\ -\mathcal{A}^T\mathcal{G}^{-1} & \mathcal{A}^T + \mathcal{A}^T\mathcal{G}^{-1}\mathcal{C} & \mathbf{1}' + \mathcal{A}^T\mathcal{G}^{-1}\mathcal{A} \end{pmatrix}, \tag{2.14}$$

where $\mathrm{a}_{MN} = \mathcal{A}^I{}_M\mathcal{A}^I{}_N$ while $\mathcal{C}_{MN}$ and $\mathbf{1}'$ are defined as above. By construction, M^{ab} is symmetric, and it has a fixed trace, $L_{ab}M^{ab} = 16$. Further, it is straightforward to verify that $M^a{}_b = M^{ac}L_{cb}$ is an element of $\mathrm{O}(d,d+16)$. In fact [23], one finds that this moduli matrix parameterizes the coset $\mathrm{O}(d,d+16)/\mathrm{O}(d) \times \mathrm{O}(d+16)$ — as a quick check, one may verify that the number of scalars, $d(d+16)$, coincides with the number of parameters in the coset. Under the duality transformations, these scalars transform as: $M^{ab} \to M'^{ab} = \Omega^a{}_c\Omega^b{}_d M^{cd}$.

With the definitions above, we can rewrite the reduced three-form field strength of (2.11) as

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2}A^a_\mu\, L_{ab}\, F^b_{\nu\lambda} + \textit{cyclic permutations} \tag{2.15}$$

where all of the gauge anomaly contributions now appear on the same footing[14]. The action of the dimensionally reduced theory (2.6) can be written in a form for which the full classical $\mathrm{O}(d,d+16)$ symmetry is manifest[24]:

$$\begin{aligned} S = \int d^Dx\sqrt{-g}e^{-\phi}\Big\{ & R + (\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\lambda} + \\ & + \frac{1}{8}L_{ab}\nabla_\mu M^{bc}L_{cd}\nabla^\mu M^{da} - \frac{1}{4}F^a_{\mu\nu}L_{ab}M^{bc}L_{cd}F^{d\,\mu\nu}\Big\}\,. \end{aligned} \tag{2.16}$$

Given the invariance of L in eq. (2.12), this action and hence the equations of motion are obviously invariant under the $\mathrm{O}(d,d+16)$ rotations: $M \to \Omega M \Omega^T$ and $F \to \Omega F$ (using matrix notation). This group includes diffeomorphisms and gauge transformations on the d-torus, which are inherited from the reduction of the metric, torsion, and gauge fields. These elements of $\mathrm{O}(d,d+16)$ essentially rearrange the components of these tensors. The nontrivial part [25, 26] of the $\mathrm{O}(d,d+16)$ group is the coset $\mathrm{O}(d,d+16)/\mathrm{O}(d) \times \mathrm{O}(d+16)$.

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Before proceeding, we comment on the origin of the $\mathrm{U}(1)^{2d+16}$ gauge symmetry of the reduced theory. The factor $\mathrm{U}(1)^{16}$ is obviously inherited from the sixteen Abelian gauge fields already present in the ten-dimensional theory. As commented above, a further $\mathrm{U}(1)^d$ comes from residual diffeomorphisms on the internal space $dy^M \to dy'^M = dy^M + d\omega^M(x)$. One might think of this as a local gauging of the translation symmetries of the internal torus. The remaining $\mathrm{U}(1)^d$ symmetry arises as a residual one-form gauge invariance with $\mathcal{B} \to \mathcal{B}' = \mathcal{B} + d\omega$ where $\omega = \omega_M(x) dy^M$. We will see below how this gauge symmetry group is modified by the introduction of axionic masses.

3. Reduction with internal fluxes

The standard Kaluza-Klein reduction, described above, can be generalized by the introduction of a constant flux of the three-form or gauge field strengths on a three- or two-cycle in the internal space. These compactifications are then similar to the Type II string and M-theory reductions considered in ref. [4]. Note that a constant internal flux requires that the corresponding potential necessarily depends on the internal coordinates. These fluxes, or alternatively the slopes for the internal dependence of the potentials, then appear as mass parameters in the reduced theory. There is also another set of masses related to certain components of the internal metric, but the discussion of these contributions is more involved and we will leave their discussion for the following section.

Using the same decomposition as in eq. (2.3), the internal fluxes are produced by including an explicit dependence on the internal coordinates in the following fields:

$$\begin{aligned}
\hat{\mathcal{A}}^I{}_M(x,y) &= \mathcal{A}^I{}_M(x) + m^I_{MN} y^N \\
\hat{\mathcal{B}}_{\mu M}(x,y) &= \mathcal{B}_{\mu M}(x) + \frac{1}{2} m^I_{MN} y^N (A^I{}_\mu(x) + \mathcal{A}^I{}_P V^P{}_\mu(x)) \\
\hat{\mathcal{B}}_{MN}(x,y) &= \mathcal{B}_{MN}(x) + \beta_{MNP} y^P + \mathcal{A}^I{}_{[M}(x) m^I_{N]P} y^P \,.
\end{aligned} \tag{3.1}$$

The other fields reduce exactly as in eq. (2.4). In this decomposition, the quantities without the caret are the local fluctuations on the base space, i.e. they are independent of the y^N. The ansatz (3.1) can be easily deduced from considering the field strengths of $\mathcal{A}^I$ and $\mathcal{B}$ on the tangent space and requiring that their fluxes on the compact space are independent of the internal coordinates. Again, the details represent the special case of the general situation discussed in the appendices B and C, when we take the internal space to be a flat d-torus without twists (i.e. with vanishing curvatures), so that the internal isometries commute. For this reason, we will not delve here into a detailed derivation of (3.1) but merely quote it. The parameters β_{MNP} comprise a totally antisymmetric constant tensor in M, N, P, and stem from the three-form fluxes on the internal torus. The constant matrices $m^I = (m^I_{MN})$ are counted by I and antisymmetric in M, N, and they correspond to the internal fluxes in the Yang-Mills sector. Note that implicitly we are following the (string theory) convention that the fields are dimensionless, and hence the flux parameters, β_{MNP} and m^I_{MN}, have the dimension of length^{-1}. Hence these parameters will appear as mass parameters in the reduced theory.

Not all m^I_{MN}'s are independent, however. As we mentioned above, the reduction ansatz must ensure that the reduced field strengths simultaneously satisfy both their equations of motion and are independent of the internal coordinates y^M. If we apply the latter demand to the tangent space components of the three-form field strength, we find that $\partial\mathcal{H}_{MNP}/\partial y^Q = 0$. This means that the form dH cannot have terms which completely reside in the compact space, as long as it is a flat torus without curvature, with the one-form basis dy^M on its tangent space. However, since $d\mathcal{H} = -\frac{1}{2}\mathcal{F}^I \wedge \mathcal{F}^I$, this translates into the condition that the Chern-Simons anomaly components must vanish in the internal space. This condition yields

$$m^I_{[MN}m^I_{PQ]} = 0\,. \tag{3.2}$$

Since m^I_{MN} are antisymmetric in the lower two indices, we can rewrite (3.2) as $m^I_{MN}m^I_{PQ} + m^I_{MP}m^I_{QN} + m^I_{MQ}m^I_{NP} = 0$. Aside from these constraints, the mass parameters m^I_{MN} are arbitrary.

In passing, we note that the ansatz (3.1) is linear in the internal coordinates despite the explicit presence of the corresponding axionic scalars without derivatives in the reduced action. One might not have expected choosing this form to succeed, given our previous results in ref. [6] on generalized axion reductions. In fact, however, the analysis there and our present results are not in conflict, as discussed in appendix E.

Using eq. (3.1), we can proceed with the dimensional reduction of the action (2.1). We will keep the discussion here as brief as possible, since this is just a special case of the more general reduction on a twisted torus that we will consider later. The best way of carrying out dimensional reduction is to follow the approach based on using gauge symmetry as the organizing principle for identifying the reduced dynamics. To do so, we use the tangent space basis defined by the zehnbein, with components

$$\mathcal{E}^A{}_M = \begin{pmatrix} e^\alpha{}_\mu & E^A{}_N V^N{}_\mu \\ 0 & E^A{}_M \end{pmatrix}\,, \qquad \mathcal{E}_A{}^M = \begin{pmatrix} e_\alpha{}^\mu & -V^M{}_\mu e_\alpha{}^\mu \\ 0 & E_A{}^M \end{pmatrix} \tag{3.3}$$

where $g_{\mu\nu} = \eta_{\alpha\beta}e^\alpha{}_\mu e^\beta{}_\nu$, $\mathcal{G}_{MN} = \delta_{AB}E^A{}_M E^B{}_N$ and $e^\alpha{}_\mu e_\alpha{}^\nu = \delta_\mu{}^\nu$ and $E^A{}_N E_A{}^M = \delta_N{}^M$, and the flat metric on the ten-dimensional tangent space is $\hat\eta_{ab} = \mathrm{diag}(\eta_{\alpha\beta}, \delta_{AB})$. To actually identify the reduced degrees of freedom, all one needs to do is to take the reduction ansatz given by eqs. (2.3) and (3.1), and compute the forms $\mathcal{R}_{\alpha\beta}$ (Riemann curvature forms), $\mathcal{F}^I$ and $\mathcal{H}$ in the "intermediate" basis spanned by dx^μ, $\mathcal{E}^A = E^A{}_N(dy^N + V^N{}_\mu dx^\mu)$. A straightforward calculation then shows that

$$\begin{aligned}\mathcal{F}^I =\ & \frac{1}{2}(F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu})dx^\mu \wedge dx^\nu + \\ & + \mathcal{D}_\mu \mathcal{A}^I{}_M E_A{}^M dx^\mu \wedge \mathcal{E}^A - m^I_{MN}E_A{}^M E_B{}^N \mathcal{E}^A \wedge \mathcal{E}^B\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\mathcal{H} = \Big(&\frac{1}{2}\partial_\mu B_{\nu\lambda} - \frac{1}{4}A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{4}B_{\mu M}V^M{}_{\nu\lambda} - \frac{1}{4}V^M{}_\mu H_{\nu\lambda M} + \\
&+ \frac{1}{4}\beta_{MNP}V^M{}_\mu V^N{}_\nu V^P{}_\lambda - \frac{1}{2}m^I_{MN}A^I{}_\mu V^M{}_\nu V^N{}_\lambda\Big) dx^\mu \wedge dx^\nu \wedge dx^\lambda + \\
&+\frac{1}{2}\Big(H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MN}V^N{}_{\mu\nu}\Big) E_A{}^M dx^\mu \wedge dx^\nu \wedge \mathcal{E}^A + \\
&+\frac{1}{2}\Big(\mathcal{D}_\mu \mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu \mathcal{A}^I{}_{N]}\Big) E_A{}^M E_B{}^N dx^\mu \wedge \mathcal{E}^A \wedge \mathcal{E}^B + \\
&+\frac{1}{6}\Big(3\beta_{MNP} + 6\mathcal{A}_{[M}m^I_{NP]}\Big) E_A{}^M E_B{}^N E_C{}^P \mathcal{E}^A \wedge \mathcal{E}^B \wedge \mathcal{E}^C ,
\end{aligned}
\tag{3.5}
$$

where in addition to eqs. (2.3), (2.4), (2.5) and (3.1), we use the following definitions:

$$
\begin{aligned}
F^I{}_{\mu\nu} &= \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu - 2m^I_{MN}V^M{}_\mu V^N{}_\nu \\
H_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP}V^N{}_\mu V^P{}_\nu + 4m^I_{MN}A^I{}_{[\mu}V^N{}_{\nu]} ,
\end{aligned}
\tag{3.6}
$$

for the reduced field strengths (while we still have $V^M{}_{\mu\nu} = \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu$) and

$$
\begin{aligned}
\mathcal{D}_\mu \mathcal{A}^I{}_M &= \partial_\mu \mathcal{A}^I{}_M - 2m^I_{MN}V^N{}_\mu \\
\mathcal{D}_\mu \mathcal{B}_{MN} &= \partial_\mu \mathcal{B}_{MN} + 2m^I_{MN}A^I{}_\mu - 3\beta_{MNP}V^P{}_\mu - 2\mathcal{A}^I{}_{[M}m^I_{N]P}V^P{}_\mu .
\end{aligned}
\tag{3.7}
$$

The expressions for the Riemann curvature can be obtained by straightforward methods. Since the metric and dilaton reduction ansatz is the same as in the conventional Kaluza-Klein case, the curvature and the dilaton make precisely the same contributions as in the previous section.

We can now reduce the action (2.1) using these results. This will generalize the Kaluza-Klein reduction of the low energy heterotic action considered in [14]. The net result of this calculation is

$$
S = S_1 + S_2 + S_3 , \tag{3.8}
$$

where the individual contributions to the action are

$$
S_1 = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 - \frac{1}{2}H^2_{\mu\nu\lambda} \Big\} , \tag{3.9}
$$

for the reduced metric-dilaton-two-form part,

$$
\begin{aligned}
S_2 = -\int d^D x \sqrt{-g} e^{-\phi} \Big\{ &W(\mathcal{G},\mathcal{A}) - \frac{1}{4}\nabla_\mu \mathcal{G}_{MN}\nabla^\mu \mathcal{G}^{MN} + \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu \mathcal{A}^I{}_M \mathcal{D}^\mu \mathcal{A}^I{}_N + \\
&+\frac{1}{4}\mathcal{G}^{MN}\mathcal{G}^{PQ}\Big(\mathcal{D}_\mu \mathcal{B}_{MP} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu \mathcal{A}^I{}_{P]}\Big)\Big(\mathcal{D}^\mu \mathcal{B}_{NQ} + \mathcal{A}^J{}_{[N}\mathcal{D}^\mu \mathcal{A}^J{}_{Q]}\Big)\Big\} ,
\end{aligned}
\tag{3.10}
$$

for the moduli fields, and

$$S_3 = -\frac{1}{4}\int d^Dx\sqrt{-g}e^{-\phi}\Big\{\mathcal{G}_{MN}V^M{}_{\mu\nu}V^{N\,\mu\nu} + f^I{}_{\mu\nu}f^{I\,\mu\nu} + \mathcal{G}^{MN}h_{\mu\nu M}h^{\mu\nu}{}_N\Big\}\,, \quad (3.11)$$

for the gauge field contributions. The auxiliary fields in eq. (3.11) are defined according to

$$\begin{aligned} f^I{}_{\mu\nu} &= F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu} \\ h_{\mu\nu M} &= H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MN}V^N{}_{\mu\nu}\,. \end{aligned} \quad (3.12)$$

The reduced three-form field strength, with all Chern-Simons contributions, is in component form

$$\begin{aligned} H_{\mu\nu\lambda} = {}& \partial_\mu B_{\nu\lambda} - \frac{1}{2}A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{2}V^M{}_\mu H_{\nu\lambda M} - \frac{1}{2}B_{\mu M}V^M{}_{\nu\lambda} + \\ &+\frac{1}{2}\beta_{MNP}V^M{}_\mu V^N{}_\nu V^P{}_\lambda - m^I_{MN}A^I{}_\mu V^M{}_\nu V^N{}_\lambda + \text{ cyclic perm.} \end{aligned} \quad (3.13)$$

In the moduli action (3.10), the function $W(\mathcal{G}_{MN}, \mathcal{A}^I{}_M)$ denotes the moduli potential, which arises because of the internal fluxes. They will in general induce an effective scalar potential, via the terms such as, e.g., $\mathcal{F}^I{}_{MN}\mathcal{F}^{I\,MN} \sim \mathcal{G}^{MN}\mathcal{G}^{PQ}m^I_{MP}m^I_{NQ}$. For the reduction scheme given by (3.1), we find that the reduced moduli potential is

$$\begin{aligned} W(\mathcal{G},\mathcal{A}) = {}& \frac{3}{4}\mathcal{G}^{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\Big(\beta_{MPR} + 2\mathcal{A}^I{}_{[M}m^I_{PR]}\Big)\Big(\beta_{NQS} + 2\mathcal{A}^J{}_{[N}m^J_{QS]}\Big) + \\ &+\mathcal{G}^{MN}\mathcal{G}^{PQ}m^I_{MP}m^I_{NQ} \end{aligned} \quad (3.14)$$

Note that this moduli potential is independent of the two-form axions $\mathcal{B}_{MN}$.

At this point we wish to rewrite the reduced action (3.8) in a more tractable form, which would highlight the symmetries of the reduced theory. In particular, we wish to determine what became of the $\mathrm{O}(d, d+16)$ symmetry which appeared in the standard Kaluza-Klein reduction. Further, the definitions (3.6) and (3.7) already suggest that the reduced gauge symmetries have become nonabelian with the internal fluxes. Below, we will confirm this with a detailed examination of the gauge symmetries of the reduced theory. Let us first try to intuitively understand how the nonabelian symmetry is produced. For this purpose, we can focus on the Yang-Mills gauge transformations. The gauge symmetries of the Kalb-Ramond gauge fields (i.e. the gauge fields arising from the reduction of the two-form) behave exactly the same. The Yang-Mills gauge transformations are the sixteen commuting symmetry transformations of the gauge fields $\mathcal{A}^I$, under which $\mathcal{A}^I \to \mathcal{A}'^I = \mathcal{A}^I + d\Lambda^I$. From the point of view of the Scherk-Schwarz reduced theory, these gauge transformations can be at most quadratic in y^M, in order not to alter the general form of the reduction ansatz (3.1). Hence, the general allowed form of gauge transformations is $\Lambda^I(x,y) = \lambda^I(x) + \lambda^I{}_M(x)y^M + \lambda^I{}_{MN}(x)y^My^N$. Here, $\lambda^I{}_{MN}(x) = \lambda^I{}_{NM}(x)$ or else it would disappear from the definition above. Further, if we turn off the Kaluza-Klein gauge fields, $\lambda^I{}_M$ and $\lambda^I{}_{MN}$ must be constant. Using the reduction ansatz for $\mathcal{A}^I$, we see that the gauge transformation properties of the fields $\mathcal{A}^I$ decompose according to

$$\mathcal{A}'^I{}_\mu = \mathcal{A}^I{}_\mu + \partial_\mu \lambda^I \,, \qquad \mathcal{A}'^I{}_M = \mathcal{A}^I{}_M + \lambda^I{}_M \,, \qquad m'^I_{MN} = m^I_{MN} + \lambda^I{}_{MN} \,. \tag{3.15}$$

Since $\lambda^I{}_M$ and $\lambda^I{}_{MN}$ are constants, we see that transformations by Λ^I have three separate effects: First, the $\lambda^I(x)$ yield the reduced version of the original $\mathrm{U}(1)^{16}$ gauge symmetry appearing in the action (2.1). Then, we obtain the global axionic symmetries, which shift the scalars $A^I{}_M$ by constants. Finally, we see that the $\lambda^I{}_{MN}$ transformations simply ensure that only the antisymmetric part of m^I_{MN} are physical. In other words, even if we had started with a general class of parameters m^I_{MN}, we can make them antisymmetric by $\lambda^I{}_{MN}$-dependent gauge transformations, leaving the rest unchanged.

The Kaluza-Klein vectors $V^M{}_\mu$ do not change under these transformation. However, due to the Chern-Simons terms, the two-form potential $\mathcal{B}$ does transform, according to

$$\delta \mathcal{B} = \frac{1}{2} \Lambda^I \mathcal{F}^I + d\Lambda \,, \tag{3.16}$$

where Λ is a residual two-form gauge transformation playing role of a custodian, which sweeps away total derivatives. It is easy to verify that the explicit form of Λ should be $\Lambda = \frac{1}{2} m^I_{MN} \lambda^I y^N dy^M - \frac{1}{2} \Lambda^I \mathcal{A}^I$. This guarantees that the reduction ansatz for the two-form in eq. (3.1) is gauge invariant. This induces nontrivial transformation properties of the fields which emerge from the two-form $\mathcal{B}$ after the reduction. Their explicit form will be given below. Here we first note that the parameters β_{MNP} change according to

$$\beta_{MNP} \to \beta'_{MNP} = \beta_{MNP} - 2\lambda^I_{[M} m^I_{N]P} \,. \tag{3.17}$$

From eq. (3.5), we see that eq. (3.15) and (3.17) combine to ensure that the internal flux of $\mathcal{H}$ remains unchanged by the axionic shifts of $A^I{}_M$.

Turning the Kaluza-Klein gauge transformations on corresponds to gauging the global axionic translations associated with the metric axions. The resulting gauge group must be nonabelian because of the anomaly in the two-form $\mathcal{B}$. The anomaly is manifest in the presence of the vector supermultiplet Chern-Simons terms in eq. (2.2). It gets enlarged further by reduction, because of the gauge anomalous decomposition of the reduced two-form $B_{\mu\nu}$.

Without further ado, we give here the final form of the infinitesimal form of the full set of reduced gauge transformations. They can be obtained by straightforward, albeit lengthy computation, of which the general case is given in appendix D. Hence the case of the flat torus, which is considered here, can be recovered as a special case. We only list the reduced vector field gauge transformations, and leave aside the one-form gauge transformations of the reduced two-form field. These reduced infinitesimal gauge transformations fall into three categories, listed here in the order of ascending complexity:

1. Kalb-Ramond gauge transformations:

$$B'_{\mu M} = B_{\mu M} + \partial_\mu \lambda_M \,, \qquad B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{2}\lambda_M V^M{}_{\mu\nu} \,. \tag{3.18}$$

2. Yang-Mills gauge transformations:

$$\begin{aligned} A'^I{}_\mu &= A^I{}_\mu + \partial_\mu \lambda^I \,, \qquad B'_{\mu M} = B_{\mu M} - 2\lambda^I m^I_{MN} V^N{}_\mu \\ B'_{MN} &= B_{MN} - 2\lambda^I m^I_{MN} \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda^I F^I{}_{\mu\nu} + m^I_{MN}\lambda^I V^M{}_\mu V^N{}_\nu \end{aligned} \tag{3.19}$$

3. Kaluza-Klein gauge transformations:

$$\begin{aligned} V'^M{}_\mu &= V^M{}_\mu + \partial_\mu \omega^M \,, \qquad A'^I{}_\mu = A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + \mathrm{O}(\omega^2) \\ B'_{\mu M} &= B_{\mu M} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + \mathrm{O}(\omega^2) \\ \mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2m^I_{MN}\omega^N \\ B'_{MN} &= B_{MN} + 3\beta_{MNP}\omega^P + 2\mathcal{A}^I{}_{[M} m^I_{N]P}\omega^P + \mathrm{O}(\omega^2) \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + \mathrm{O}(\omega^2) \end{aligned} \tag{3.20}$$

Here the fields that are not listed above explicitly are invariant under the corresponding transformations.

We can now determine the algebra of the gauge group defined by eqs. (3.18)-(3.20). Understanding the full structure of the gauge symmetry is facilitated by adopting the $\mathrm{O}(d, d+16)$ notation introduced previously, which combines the three sets of gauge fields in a single multiplet (2.13). We can then define a combined gauge transformation parameter

$$\hat{\omega}^a = (\omega^M, \lambda_M, \lambda^I) \,. \tag{3.21}$$

Now the proper way to examine the gauge symmetries is to introduce the corresponding generators T_a, where the index now takes values in the space of $\{\,_M,\ ^M,\ ^I\}$, of dimension $2d+16$. We explicitly denote the separate generators as $T_a = (Z_M, X^M, Y^I)$ — note the order: first Kaluza-Klein, then Kalb-Ramond and finally Yang-Mills. We expect that their algebra satisfies

$$[T_a, T_b] = i f_{ab}{}^c T_c \,, \tag{3.22}$$

where $f_{ab}{}^c$ are some structure constants to be determined. In order to find them, we consider the successive application of gauge transformations (3.18)–(3.20) of the form $h^{-1} \cdot g^{-1} \cdot h \cdot g$ where h and g are two of any of the three above types of gauge transformations. We can use the Baker-Campbell-Hausdorff formula to determine the product. Infinitesimally, for any two operators A, B and a number α we have $e^{\alpha A} B e^{-\alpha A} = B + \alpha[A, B] + O(\alpha^2)$ where $[\ ,\ ]$ denotes the commutator. If $g = \exp(i\hat{\omega}_1^a T_a)$ and $h = \exp(i\hat{\omega}_2^a T_a)$, with gauge parameters $\hat{\omega}_{1,2}^a$ as in eq. (3.21), we have, to the lowest order in ω_1^a, $g^{-1} \cdot h \cdot g = h - i\hat{\omega}_1^a [T_a, h]$ and hence we find that, to lowest order:

$$h^{-1} \cdot g^{-1} \cdot h \cdot g = 1 + \hat{\omega}_1^a \, \hat{\omega}_2^b \, [T_a, T_b] = 1 + i f_{ab}{}^c \, \hat{\omega}_1^a \, \hat{\omega}_2^b \, T_c \, . \tag{3.23}$$

Substituting the explicit form of the reduced gauge transformations (3.18)–(3.20), we can deduce the structure constants. Since they do not depend on a representation, we need to project the operator expression (3.23) on a set of basis vectors of a faithful irreducible representation of the gauge group. In other words, we should evaluate $h^{-1} \cdot g^{-1} \cdot h \cdot g|\Psi\rangle$ for a set of basis vectors $|\Psi\rangle$. Vector fields must provide a faithful representation, and hence using (3.18)–(3.20), we find the following equations for the structure constants:

$$f^I{}_{MN} = f_{MN}{}^I = 2m^I_{MN} \, , \qquad f_{MNP} = -3\beta_{MNP} \, . \tag{3.24}$$

These equations must hold in order to get the rules for the combination of gauge transformation functions, $\lambda_M = \omega^N \lambda^I f^I{}_{NM} = -2m^I_{MN} \omega^N \lambda^I$ and $\lambda_P = f_{MNP} \omega_1^M \omega_2^N = 3\beta_{PNM} \omega_1^M \omega_2^N$, as can be seen from the comparison of the compositions with the original gauge transformations. Hence finally, the gauge algebra is

$$\begin{aligned} [X^M, X^N] &= [Y^I, Y^J] = [X^M, Y^I] = [X^M, Z_N] = 0 \\ [Y^I, Z_M] &= 2im^I_{MN} X^N \\ [Z_M, Z_N] &= -3i\beta_{MNP} X^P + 2im^I_{MN} Y^I \end{aligned} \tag{3.25}$$

Thus we see that the reduced gauge group is indeed nonabelian with the internal flux parameters playing the role of the structure constants.

In order to rewrite the reduced theory in the form where the nonabelian gauge symmetry is manifest, we first need to define a metric on the Lie algebra of the gauge group. Normally, one would use the standard Cartan metric on a Lie Algebra, given by $C_{ab} = f_{ac}{}^d f_{bd}{}^c$. However, from eq. (3.25) in our example the gauge algebra is not semi-simple. This implies that the Cartan metric is degenerate, and so without an inverse. By comparing the gauge algebra with the algebra of the $O(d, d+16)$ duality group which arises in the reduced action, one can verify that the gauge algebra has become embedded in the duality algebra. The reason is that the nonabelian gauge transformations, as dictated by eq. (3.25), now mix the gauge generators, a function previously reserved solely for the duality transformations. However, even without this observation, and by considering the case of the standard massless Kaluza-Klein reduction from the previous section, we see that the natural metric in the gauge algebra can be defined by $\langle T_a, T_b \rangle \equiv L_{ab}$. We will return to this interplay of the gauge and U-duality group later.

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Identifying our Lie-algebra-valued one-form gauge potential as

$$A = A^a T_a = V^M Z_M + B_M X^M + A^I Y^I \,, \tag{3.26}$$

we find that the corresponding nonabelian field strength

$$F = dA + iA \wedge A = \frac{1}{2} F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu \tag{3.27}$$

has components which coincide with the expressions for field strengths (3.6) that come from dimensional reduction:

$$F^a_{\mu\nu} = ({V^M}_{\mu\nu},\ H_{\mu\nu M},\ {F^I}_{\mu\nu})\,. \tag{3.28}$$

In component form, we have

$$\begin{aligned}
{F_{\mu\nu}}^M &= \partial_\mu {V^M}_\nu - \partial_\nu {V^M}_\mu \\
{F_{\mu\nu}}^I &= \partial_\mu {A^I}_\nu - \partial_\nu {A^I}_\mu - 2m^I_{MN} {V^M}_\mu {V^N}_\nu \\
F_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 4m^I_{MN} {A^I}_{[\mu} {V^N}_{\nu]} + 3\beta_{MNP} {V^N}_\mu {V^P}_\nu \,.
\end{aligned} \tag{3.29}$$

The coupling constant may appear to be normalized to unity in eq. (3.27). More accurately, the coupling is absorbed into the structure constants (3.24), as the flux parameters are by the present analysis arbitrary in magnitude.

We can also compute the Chern-Simons form for the gauge field (3.26). The result is

$$\begin{aligned}
\omega_{CS} &= \langle A \wedge F - \frac{i}{3} A \wedge A \wedge A \rangle \\
&= \frac{1}{3} dx^\mu \wedge dx^\nu \wedge dx^\lambda \left(\frac{1}{2} {A^I}_\mu {F^I}_{\nu\lambda} + \frac{1}{2} {V^M}_\mu H_{\nu\lambda M} + \frac{1}{2} B_{\mu M} {V^M}_{\nu\lambda} + \right. \\
&\quad \left. + \frac{1}{2} \beta_{MNP} {V^M}_\mu {V^N}_\nu {V^P}_\lambda - m^I_{MN} {A^I}_\mu {V^M}_\nu {V^N}_\lambda + \text{cyclic perm.} \right),
\end{aligned} \tag{3.30}$$

which corresponds to precisely the anomaly contribution in the reduced three-form field strength, obtained by dimensional reduction, and given in eq. (3.13). Hence, we see that with the internal fluxes all of the Chern-Simons contributions terms in eq. (3.13) can be organized into a single nonabelian structure, and that, in form notation, the reduced three-form field strength is then simply

$$H = dB - \frac{1}{2} \omega_{CS} \,. \tag{3.31}$$

This is exactly what is needed to maintain simultaneously U-duality and gauge invariance. The transformation formulas for $B_{\mu\nu}$ always contain the anomalous gauge transformation of B. This is essential to establish the $\mathrm{O}(d, d+16)$ duality invariance of the reduced theory, as shown by Maharana and Schwarz [14]. However, in the case of the Scherk-Schwarz dimensional reduction, there are additional terms arising in order to account for additional nonlinearities present in the Chern-Simons sector, and we have seen here that these are precisely in the form of the nonabelian anomalous contributions.

Finally, we can put the moduli potential (3.14) in a symmetric form. We lower the last index of the structure constants to define the completely antisymmetric tensor in the Lie group $f_{abc} = f_{[ab}{}^{d}L_{d|c]}$. Using this tensor and the moduli matrix M^{ab} introduced in eq. (2.14), we can write the potential for these scalars as:

$$W(M) = W(\mathcal{G},\mathcal{B},\mathcal{A}) = \frac{1}{12}M^{ad}M^{be}M^{cf}f_{abc}f_{def}\,. \tag{3.32}$$

Further, the constraint (3.2) for the masses $m^{I}_{[AB}m^{I}_{CD]} = 0$ is encoded in

$$L^{ad}f_{a[bc|}f_{d|ef]} = \frac{1}{3}L^{ad}\Big(f_{abc}f_{def} + f_{abe}f_{dfc} + f_{abf}f_{dce}\Big) = 0\,. \tag{3.33}$$

That is, the flux constraint is precisely the only nontrivial component of the Jacobi identity for the Lie algebra defined with the structure constants $f_{ab}{}^{c}$ as in eq. (3.24) and with the metric L_{ab}.

Armed with these expressions, we can write down the reduced action in the manifestly covariant form. Having squared away the anomaly, we note that the gauge kinetic terms in the reduced action (3.11) can be rewritten as

$$\mathcal{F}^2 = F^{a}_{\mu\nu}L_{ab}M^{bc}L_{cd}F^{d\mu\nu} \tag{3.34}$$

precisely as before, although the field strengths (3.28) are now nonabelian. The nonabelian field strengths vary under gauge transformations according to

$$F_{\mu\nu} \to F'_{\mu\nu} = U\cdot F_{\mu\nu}\cdot U^{-1} = F^{a}_{\mu\nu}U\cdot T_a\cdot U^{-1} = U^{b}{}_{a}F^{a}_{\mu\nu}\,T_b\,. \tag{3.35}$$

Hence, the transformation law for the gauge field strength is $F'^{a}_{\mu\nu} = U^{a}{}_{b}F^{b}_{\mu\nu}$. Gauge invariance of the kinetic terms thus demands that the matrix M must also transform nontrivially under the nonabelian group, in contrast to the simple Abelian situation encountered in standard Kaluza-Klein reduction. Indeed, we find

$$M_{ab} \to M'_{ab} = U^{c}{}_{a}U^{d}{}_{b}M_{cd}\,, \qquad M^{ab} \to M'^{ab} = U^{a}{}_{c}U^{b}{}_{d}M^{cd}\,, \tag{3.36}$$

where $M_{ab} \equiv L_{ac}M^{cd}L_{db}$. The second transformation is derived by noting that the gauge transformation $U^{a}{}_{b}$ is a $\mathrm{O}(d,d+16)$ matrix. Hence since F^a transform as an adjoint of the gauge group, M^{ab} transforms as a (symmetrized) product of the two adjoints. Therefore, the kinetic terms for the moduli fields must contain gauge covariant derivatives

$$\mathcal{D}_\mu M^{ab} = \partial_\mu M^{ab} - f_{cd}{}^{a}A^{c}_{\mu}M_{db} - f_{cd}{}^{b}A^{c}_{\mu}M^{ad}\,. \tag{3.37}$$

Combining the various expressions above, one finds that these expressions are reproduced precisely by the derivatives (3.7) appearing in the scalar kinetic terms in eq. (3.10).

The reduced action can therefore be put in a U-duality invariant form:

$$S = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 + \frac{1}{8} L_{ab} \mathcal{D}_\mu M^{bc} L_{cd} \mathcal{D}^\mu M^{da} - \\ - \frac{1}{4} F^a_{\mu\nu} L_{ab} M^{bc} L_{cd} F^{d\mu\nu} - \frac{1}{12} H^2_{\mu\nu\lambda} - W(M) \Big\} . \quad (3.38)$$

Hence the generalized reduction has produced an effective D-dimensional action of a form very similar to that (2.16) produced by the standard Kaluza-Klein reduction. The main differences arise in the appearance of a nonabelian gauge symmetry and the scalar potential $W(M)$. The parameters introduced by the internal fluxes play a dual role as structure constants in the nonabelian group, and mass couplings determining the potential for the moduli. With respect to the duality symmetry, it may appear from the form of the action (3.38) that the $\mathrm{O}(d, d + 16)$ symmetry is still present but in a slightly broken form, due to the constants f_{abc}. An alternate point of view, which we advocate, is that $\mathrm{O}(d, d + 16)$ is a symmetry of the reduced theory, and that we are discovering here that the usual transformations, $M \to M' = \Omega M \Omega^T$ and $F \to F' = \Omega F$, must be supplemented by a transformation of the fluxes:

$$f_{abc} \to f'_{abc} = \Omega_a{}^d \Omega_b{}^e \Omega_c{}^f f_{def} , \quad (3.39)$$

where in the present notation, $\Omega_a{}^b = L_{ac} \Omega^c{}_d L^{db} = (\Omega^{-1})^b{}_a$. In the following sections, we will find that similar results apply for the generalized reduction when mass parameters are also introduced through the metric axions.

4. Twisted tori

We now turn to the question of inducing masses in general. In particular, we wish to include mass parameters arising from internal coordinate dependence of the metric. Metric axion masses may be associated with axionic symmetries for internal metric components, and so an easy way to identify these is to first reduce the theory by the standard Kaluza-Klein procedure and then look for axionic symmetries in the reduced theory. In toroidal standard reductions, a metric axion first appears from the off-diagonal component of the internal metric in compactifying on a two-torus. Hence a mass parameter can be generated by introducing an internal coordinate dependence in this axion in a further S^1 reduction. Geometrically, the axionic symmetry above corresponds to a constant shift in the modular parameter of the two-torus. The subsequent reduction then produces a nontrivial $\mathrm{SL}(2,\mathbb{R})$ fibre bundle with T^2 fibre over S^1. That is the T^2 geometry varies in traversing the S^1, and in one cycle, its metric only returns to its original form up to an $\mathrm{SL}(2,\mathbb{R})$ transformation. Thus the resulting geometry is similar to that appearing around a stringy cosmic string [31], whose construction relies on the same axionic symmetry of the two-torus reduction.

This geometry was also recently discussed[8, 9] in relating M-theory to massive IIa string theory. We will refer to this three-torus with such nontrivial curvature and its higher dimensional generalizations as "twisted tori".

A sequential dimensional descent extending the T^3 reduction described above may not be the most useful tool to derive the most general array of metric mass parameters. Further the geometric picture does not give us a clear picture of the most general compactification. In the following, we will consider a generalized ansatz in which the internal components of the metric can have complicated dependence on internal coordinates, restricted only by the requirement that the internal manifold has an isometry group acting transitively on it [34]. Indeed, as we will see, the general procedure to turn on all available metric mass terms is to perform dimensional reduction on all nonabelian isometry groups consistent with the dimension of the compact space. That is, while throughout this section, we have in mind the twisted tori described above, the formalism developed here and the subsequent results apply for more general compactifications. The essential ingredient is an isometry group with a transitive action and with the same dimension as the internal space. Hence the topology of the internal manifold is not restricted to be homeomorphic to fibered tori, and we will consider explicit examples in section 6.

Our construction is motivated by the general considerations of [5], and a concrete example provided in ref. [19]. The compactification considered there may be interpreted as a twisted three-torus. Explicitly, the basis one-forms on this torus were found to be $\eta^1 = dx + Q(ydz - zdy), \eta^2 = dy, \eta^3 = dz$. This basis contains a noncommutative structure, as can be seen from the fact that $d\eta^1 = 2Q\eta^2 \wedge \eta^3$. This in turn resulted in a nonabelian isometry group of the internal space, with the nontrivial commutator between the dual Killing vectors: $[Z_2, Z_3] = -2QZ_1$. This algebra is known from the studies of homogeneous anisotropic spaces under the name of Bianchi II algebra. The vector fields Z_M are generators of generalized translations, which leave the geometry invariant. Hence the notion of homogeneity.

Our generalized ansatz builds on the above result with the introduction of tangent space basis forms on the internal manifold of the form

$$\mathcal{E}^A = \mathcal{E}^A{}_M(x)\, \eta^M(y, dy)\,, \tag{4.1}$$

where the one-forms η^M are allowed to depend on the internal coordinates, in such a way that they yield all allowed isometry groups with the dimension of the Lie algebra equal to the given dimension of the internal space. The Killing isometries generalize the internal space translations present in the simplest reductions on flat tori. The corresponding Killing vector fields Z_M, which generate these isometries, are dual to the basis one-forms η^M, i.e. $Z_M(\eta^N) = \delta^M_N$. To bring the full force of symmetry to good use, we define the pullback functions $N_N{}^M(y)$ from the holonomic basis dy^M to the twisted basis η^M:

$$\eta^M = N_N{}^M(y) dy^N\,. \tag{4.2}$$

Since the matrix $N_N{}^M(y)$ is a map between two bases, it must be everywhere invertible on a smooth manifold. Let its inverse be $N^M{}_N(y)$, so that the inverse map to (4.2) is $dy^M = N^M{}_N(y)\eta^N$. For the simple reduction in the Bianchi II example, both $N_N{}^M(y)$ and its inverse depended linearly on y^M.

The Killing vector fields can be expressed in terms of the holonomic coordinate vector fields $\partial_M = \partial/\partial y^M$ and the inverse pullback matrix $N^M{}_N(y)$ as

$$Z_M = N^N{}_M(y)\partial_N\,. \tag{4.3}$$

They must satisfy a Lie algebra specified by a set of fixed structure constants $2\gamma^M_{NP}$

$$[Z_M, Z_N] = 2\gamma^P_{MN} Z_P\,, \tag{4.4}$$

where we have introduced the factor of 2 for later convenience. The dual map between η^M and Z_M then implies[2]

$$d\eta^M = -\gamma^M_{NP}\eta^N \wedge \eta^P\,. \tag{4.5}$$

By the Bianchi identity for basis one-forms, $d^2\eta^N = 0$, we see that the structure constants must satisfy

$$\gamma^M_{R[N}\,\gamma^R_{PQ]} = 0\,, \tag{4.6}$$

where the square brackets denote antisymmetrization of the enclosed indices. This identity must hold for any set of values of M, N, P, Q. Hence, the symmetry algebra is encoded in the commutation relations of Killing isometries (4.4) and derivatives of their dual one-forms (4.5). The structure constants γ^M_{NP} of the isometry group on the internal space will play the role of axionic masses in the reduced theory. This is evident from the simple Bianchi II example considered in ref. [19].

A final constraint on γ^M_{NP} can be obtained by considering the volume form on the internal space, $V_d = \frac{1}{d!}\epsilon_{N_1\ldots N_d}\eta^{N_1}\wedge\ldots\wedge\eta^{N_d} = \det(N_N{}^M)d^dy$. Invariance of this measure under the Killing isometries Z_M implies [5]

$$\gamma^N_{NM} = 0\,. \tag{4.7}$$

We can see this as follows: We require that $\mathcal{L}_{Z_M}V_d = 0$ where $\mathcal{L}$ denotes the Lie derivative. For differential forms, one has $\mathcal{L}_\xi = di_\xi + i_\xi d$ where i_ξ denotes the interior product with a vector ξ. Further in the case of interest $dV_d = 0$ since it is the top form on the internal space, and so

$$\mathcal{L}_{Z_M}V_d = di_{Z_M}V_d = (-)^{M+1}d\left(\eta^1\wedge\ldots\wedge\widehat{\eta}^M\wedge\ldots\wedge\eta^d\right), \tag{4.8}$$

[2] This can be seen as follows. From $\mathcal{L}_{Z_M}(Z_N(\eta^P)) = \mathcal{L}_{Z_M}\delta^M_N = 0$ we have $Z_N(\mathcal{L}_{Z_M}\eta^P) = -[Z_M, Z_N](\eta^P) = -2\gamma^Q_{MN}Z_Q(\eta^P) = -2\gamma^P_{MN}$. Next, $\mathcal{L}_{Z_M}\eta^P = d(Z_M(\eta^P)) + Z_M(d\eta^P)$. Therefore, if we expand the two-form $d\eta^P$ in terms of the basis forms, $d\eta^P = \eta^P_{MN}\eta^M\wedge\eta^N$, we find that $Z_N(\mathcal{L}_{Z_M}\eta^P) = 2\eta^P_{MN}$. Thus we arrive at $\eta^P_{MN} = -\gamma^P_{MN}$, and $d\eta^P = -\gamma^P_{MN}\eta^M\wedge\eta^N$, as claimed.

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where the caret denotes that η^M does not appear in the wedge product. Given the exterior derivatives of the basis forms in (4.5), it is not hard to see that $\mathcal{L}_{Z_M} V_d = \gamma^N_{NM} V_d$, and so invariance requires eq. (4.7).

Let us now consider the infinitesimal form of the isometries, which we determine from the Lie algebra. Since an infinitesimal transformation of a tensor field T generated by a Killing field $Z = \omega^P Z_P$ can be written as $\delta T = \mathcal{L}_Z T$, we see that the basis 1-forms transform according to

$$\eta^N \to (\eta')^N = \mathcal{S}^N{}_M \eta^M \,, \tag{4.9}$$

where the matrix $\mathcal{S}^M{}_N(\omega)$ belongs to the adjoint of the group of diffeomorphisms on the internal space defined by (4.4). Hence, this matrix can be written in infinitesimal form as

$$\mathcal{S}^M{}_N(\omega) = \delta^M_N - 2\gamma^M_{NP}\omega^P(x) \,. \tag{4.10}$$

It will be useful to also define the matrix $\mathcal{O}^M{}_N(\omega)$, as formally a square root of $\mathcal{S}$. In infinitesimal form,

$$\mathcal{O}^M{}_N(\omega) = \delta^M_N - \gamma^M_{NP}\omega^P(x) \,. \tag{4.11}$$

This matrix will encode the gauge transformations of the gauge fields after dimensional reduction. To see this, we can take the parameters $\omega^M(x)$ to depend on the base space coordinates, in which case the transformations of the basis 1-forms η^M obey

$$\eta'^M = \mathcal{S}^M{}_N \eta^N - \mathcal{O}^M{}_N d\omega^N \,, \tag{4.12}$$

where $d\omega^N = \partial_\mu \omega^N dx^\mu$ takes into account the fact that ω^N are defined as local functions on the reduced base space. The factor of 2 in $S^M{}_N$ comes about because the charge of η^M is 1, and the structure constants are $2\gamma^M_{NP}$.

For completeness we note that we can define the invariant forms under the action of the symmetry group (4.10). The simplest way to construct them is to first look for invariant vector fields, and then determine their duals. The invariant vector fields are defined by $\mathcal{L}_{Z_M}\mathcal{Y} = 0$ for all Z_M. Writing out this condition explicitly, we see that the most general solution is

$$\mathcal{Y} = \mathcal{X}^N N_N{}^M \partial_M = \mathcal{X}^M \mathcal{Y}_M \,, \tag{4.13}$$

where $\mathcal{X}^M$ are arbitrary constants. Hence the maximal set of vector fields invariant under (4.10) is the set of $\mathcal{Y}_M = N_M{}^N \partial_N$. Their dual one-forms

$$\zeta^M = N^M{}_N dy^N \tag{4.14}$$

are also invariant under the isometry group, $\mathcal{L}_{Z_M}\zeta^N = 0$. Note that in the terminology of group manifolds the forms ζ^M correspond to the right-invariant Maurer-Cartan forms on the group space, whilst our basis 1-forms η^M correspond to the left-invariant

Maurer-Cartan forms. Having chosen η^M as the basis 1-forms, we have elected to express the internal metric in terms of the left-invariant forms. One might be tempted to use the forms ζ^M for the dimensional reduction of invariant objects, but encoding the isometries with the use of the left-invariant generators Z_M. However, as we will discuss below, this is not the optimal starting point for dimensional reduction in our case, since it is incapable of recording axionic shift symmetries, necessary to turn the axionic masses on. As it turns out, this can be accomplished with the use of the basis one-forms η^M which are dual to the group generators.

At this point we need to define the proper ansatz for the dimensional reduction with the group of isometries as defined by eqs. (4.4) and (4.5). This is a subtle issue due to the large number of degrees of freedom and the interplay of different symmetries present in the model. The ansatz for the reduction must correctly convert the original symmetries of the compactification manifold into the reduced gauge symmetries, while simultaneously ensuring the equivalence of the dynamics. In other words, the ansatz must be consistent with the ten-dimensional equations of motion in that the variations of the reduced action produce the same set of equations. We begin by considering first the reduction ansatz for the metric. The choice of Z_M's as the symmetry generators implies that the consistent ansatz for the nonabelian Kaluza-Klein reduction where the symmetry group is defined by eqs. (4.4) and (4.5) is found by expressing the metric on the internal manifold as a bilinear combination of the basis one-forms η^M. When the group is gauged, and the forms η^M transform according to (4.12), we must introduce the Kaluza-Klein gauge fields $V^M{}_\mu$ to compensate the $d\omega^M$ terms in (4.12). It is not hard to see that this implies the following ansatz for the reduction of the metric:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \mathcal{G}_{MN}(x)(\eta^M(y) + V^M{}_\mu(x)dx^\mu)(\eta^N(y) + V^N{}_\nu(x)dx^\nu)\,. \quad (4.15)$$

The matrix $\mathcal{G}_{MN}(x)$ contains the desired axionic degrees of freedom, which can be seen as follows: first, the transformation rule for η^M in (4.12) requires that the vector field $V^M{}_\mu$ transforms according to

$$V'^M{}_\mu = \mathcal{S}^M{}_N V^N{}_\mu + \mathcal{O}^M{}_N \partial_\mu \omega^N \quad (4.16)$$

and that the matrix $\mathcal{G}_{MN}$ transforms according to

$$\mathcal{G}'_{MN} = \mathcal{S}_M{}^P \mathcal{S}_N{}^Q \mathcal{G}_{PQ}\,, \quad (4.17)$$

where $\mathcal{S}_M{}^N$ is the inverse matrix of $\mathcal{S}^M{}_N$. If we decompose the matrix $\mathcal{G}_{MN}$ in terms of the d-bein $\mathcal{E}^A{}_M$, defined by $\mathcal{G}_{MN} = \delta_{AB}\mathcal{E}^A{}_M\mathcal{E}^B{}_N$, $\mathcal{E}^A{}_N\mathcal{E}_B{}^N = \delta^A_B$ and $\mathcal{E}^A{}_N\mathcal{E}_A{}^M = \delta^M_N$, we can see that the d-bein transform according to

$$\mathcal{E}'^A{}_N = \mathcal{S}_N{}^P \mathcal{E}^A{}_P\,, \quad (4.18)$$

where $\mathcal{S}_N{}^P$ is the inverse of the matrix (4.10). In the infinitesimal form, to linear order in ω^P, this equation becomes

$$\mathcal{E}'^A{}_N = \mathcal{E}^A{}_N + 2\mathcal{E}^A{}_P\,\gamma^P_{NM}\,\omega^M\,. \tag{4.19}$$

It identifies the d-bein as an array of scalars belonging to both the singlet and the adjoint (with charge -1) representations of the isometry group. As we alluded to at the beginning of this section, this is precisely the transformation rule needed for extracting the axionic degrees of freedom. They can be explicitly found by carrying out the Gauss decomposition of the matrix $\mathcal{E}^M{}_N$ and identifying the pivots with the dilatonic degrees of freedom and the upper triangular matrix elements with the axions. For example on T^3, one has

$$\begin{aligned}\mathcal{E}^A{}_M &= \begin{pmatrix} e^{\phi_1} & e^{\phi_1}\mathcal{A}_0^{(12)}(y) & e^{\phi_1}\mathcal{A}_0^{(13)}(y) \\ 0 & e^{\phi_2} & e^{\phi_2}\mathcal{A}_0^{(23)}(y) \\ 0 & 0 & e^{\phi_3}\end{pmatrix} \\ &= \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{\phi_2} & 0 \\ 0 & 0 & e^{\phi_3}\end{pmatrix}\begin{pmatrix} 1 & \mathcal{A}_0^{(12)}(y) & \mathcal{A}_0^{(13)}(y) \\ 0 & 1 & \mathcal{A}_0^{(23)}(y) \\ 0 & 0 & 1\end{pmatrix},\end{aligned} \tag{4.20}$$

with the dilatons ϕ_i and the axions $\mathcal{A}_0^{(ij)}$. We will proceed with the dimensional reduction on the metric (4.15) in the next section. Here we merely pause to note that the tangent space basis forms can be expressed in terms of $e^\alpha = e^\alpha{}_\mu dx^\mu$ and

$$\mathcal{E}^A = \mathcal{E}^A{}_N(\eta^N + V^N{}_\mu dx^\mu)\,, \tag{4.21}$$

which are invariant under the isometry group, as can be seen from combining (4.12), (4.16) and (4.19).

We now sketch the Ansätze for the simultaneous reduction of the vector fields $\mathcal{A}^I$ and the two-form potential $\mathcal{B}$ consistent with the above metric Ansätze — see Appendices B and C for the complete details. It is clear that the vector fields must be expressed as

$$\mathcal{A}^I = \mathcal{A}^I{}_\mu(x)dx^\mu + \mathcal{A}^I{}_M(x)\eta^M + \sigma^I(y)\,, \tag{4.22}$$

where the forms σ^I are to be determined shortly. The reason for this form of the ansatz is that it ensures the correct transformation rules for $\mathcal{A}^I{}_M(x)$ after the reduction, so that it retains its role as the axion of the reduced theory. If we had attempted to reduce the gauge field $\mathcal{A}^I$ such that this term were replaced by $\bar{\mathcal{A}}^I{}_M\zeta^M$, where ζ^M were the invariant one-forms defined in (4.14), the reduced quantity $\bar{\mathcal{A}}^I{}_M$ would have been gauge singlets, and hence would not have transformed as given in (3.20) in the limit $\gamma^M_{LN} = 0$. Eq. (4.22) is the only possibility for reduction which could reduce to the correct limit as defined in (3.20). To determine σ^I, we turn to the gauge field strength $\mathcal{F}^I = d\mathcal{A}^I$. Direct evaluation gives

$$\mathcal{F}^I = \partial_{[\mu}\mathcal{A}^I{}_{\nu]}dx^\mu\wedge dx^\nu + \partial_\mu\mathcal{A}^I{}_M\eta^M - \mathcal{A}^I{}_M\gamma^M_{NP}\eta^N\wedge\eta^P + d\sigma^I\,. \tag{4.23}$$

However by requiring the covariance of the reduced theory we must have

$$\mathcal{F}^I = \partial_{[\mu}\mathcal{A}^I{}_{\nu]}dx^\mu \wedge dx^\nu + \partial_\mu \mathcal{A}^I{}_M \eta^M - (m^I_{NP} + \mathcal{A}^I{}_M \gamma^M_{NP})\eta^N \wedge \eta^P\,, \tag{4.24}$$

which formally coincides with (3.4) in the limit $\gamma^M_{NP} = 0$. Note that this expression must be valid in general, for any internal isometry group. Here m^I_{NP} are constants which are antisymmetric in the lower two indices, and in the limit $\gamma^M_{NP} = 0$ they must become identical with the axionic Yang-Mills masses discussed in the previous section. In terms of the m^I_{MN}, we find that

$$d\sigma^I = -m^I_{MN}\eta^M \wedge \eta^N\,. \tag{4.25}$$

The integrability condition for (4.25) gives $m^I_{Q[M}\gamma^Q_{NP]}\eta^M \wedge \eta^N \wedge \eta^P = 0$, which leads to

$$m^I_{MQ}\gamma^Q_{NP} + m^I_{NQ}\gamma^Q_{PM} + m^I_{PQ}\gamma^Q_{MN} = 0\,, \tag{4.26}$$

which, as we will see, will turn out to be exactly one of the Jacobi identities for the structure constants of the reduced theory. Comparing (4.22) with the ansatz for $\hat{\mathcal{A}}^I(x,y)$ in (3.1) shows that they coincide when $\gamma^M_{NP} = 0$, while the constraint (4.26) disappears. Hence (4.22) and the solution of (4.25) together comprise the reduction ansatz for the Yang-Mills gauge fields.

The last remaining ingredient for the reduction ansatz is the two-form potential $\mathcal{B}$. We can always write $\mathcal{B}$ in the form given in (2.3). We have to assign the y^M dependence to the components of $\mathcal{B}$ in such a way as to ensure that the three-form field strength $\mathcal{H}$ is independent of y^M if expanded in the dx^μ, η^M, or equivalently, $e^\alpha, \mathcal{E}^M$ basis. The former basis is much more convenient for calculational simplicity, and we use it here. In this basis,

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{6}\mathcal{H}_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda + \frac{1}{2}\mathcal{H}_{\mu\nu M}dx^\mu \wedge dx^\nu \wedge \eta^M + \\ & + \frac{1}{2}\mathcal{H}_{\mu MN}dx^\mu \wedge \eta^M \wedge \eta^N + \frac{1}{6}\mathcal{H}_{MNP}\eta^M \wedge \eta^N \wedge \eta^P\end{aligned} \tag{4.27}$$

On the other hand, $\mathcal{H} = d\mathcal{B} - \frac{1}{2}\mathcal{A}^I \wedge \mathcal{F}^I$. We can evaluate the Chern-Simons contribution, using (4.22) and (4.24), to find

$$\begin{aligned}\mathcal{A}^I \wedge \mathcal{F}^I = {} & \frac{1}{2}\mathcal{A}^I{}_\mu \mathcal{F}^I{}_{\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda + \\ & + \left(\frac{1}{2}\mathcal{F}^I{}_{\mu\nu}(\mathcal{A}^I{}_M + \sigma^I{}_M) + \mathcal{A}^I{}_\mu \partial_\nu \mathcal{A}^I{}_M\right)dx^\mu \wedge dx^\nu \wedge \eta^M + \\ & + \left(\partial_\mu \mathcal{A}^I{}_M\left(\mathcal{A}^I{}_N + \sigma^I{}_N\right) - \mathcal{A}^I{}_\mu\left(m^I_{MN} + \mathcal{A}^I{}_P\gamma^P_{MN}\right)\right)dx^\mu \wedge \eta^M \wedge \eta^N + \\ & - \left(\mathcal{A}^I{}_M + \sigma^I{}_M\right)\left(m^I_{NP} + \mathcal{A}^I{}_Q\gamma^Q_{NP}\right)\eta^M \wedge \eta^N \wedge \eta^P\,.\end{aligned} \tag{4.28}$$

Since this expression contains the terms which depend explicitly on y^M's, we must choose the ansatz for $\mathcal{B}$ such that this dependence cancels, and the components of $\mathcal{H}$ in the dx^μ, η^M basis depend explicitly only on x^μ. With this ansatz, the field strength becomes

$$\begin{aligned}\mathcal{H} = & \left(\frac{1}{2}\partial_{[\mu}\mathcal{B}_{\nu\lambda]} - \frac{1}{4}\mathcal{A}^I{}_\mu\mathcal{F}^I{}_{\nu\lambda}\right)dx^\mu \wedge dx^\nu \wedge dx^\lambda + \\ & + \left(\partial_{[\mu}\mathcal{B}_{\nu]M} - \frac{1}{2}(\mathcal{F}^I{}_{\mu\nu}\mathcal{A}^I{}_M + \mathcal{A}^I{}_\mu\partial_\nu\mathcal{A}^I{}_M)\right)dx^\mu \wedge dx^\nu \wedge \eta^M + \\ & + \frac{1}{2}\left(\partial_\mu\mathcal{B}_{MN} - \mathcal{B}_{\mu P}\gamma^P_{MN} + \frac{1}{2}\left(\mathcal{A}^I{}_M\partial_\mu\mathcal{A}^I{}_N + \mathcal{A}^I{}_\mu(2m^I_{MN} + \mathcal{A}^I{}_P\gamma^P_{MN})\right)\right)dx^\mu \wedge \eta^M \wedge \eta^N + \\ & + \frac{1}{2}\mathcal{A}^I{}_M\left(2m^I_{NP} + \mathcal{A}^I{}_Q\gamma^Q_{NP}\right)\eta^M \wedge \eta^N \wedge \eta^P + \frac{1}{2}\beta_{MNP}\eta^M \wedge \eta^N \wedge \eta^P \end{aligned} \tag{4.29}$$

The constants β_{MNP} are the axionic masses for the two-form, which become identical to their corresponding parameters discussed in the previous section when $\gamma^M_{NP} = 0$. Here we also have an additional constraint. The integrability condition for the field strength, which follows from its definition, is $d\mathcal{H} = -\frac{1}{2}\mathcal{F}^I \wedge \mathcal{F}^I$. Substituting in this equation (4.29) and (4.24), we find the following expression relating β_{MNP}, γ^M_{NP}, and m^I_{MN}:

$$3\beta_{L[MN}\gamma^L_{PQ]} = m^I_{[MN}m^I_{PQ]} \tag{4.30}$$

This relation represents the last of the Jacobi identities for the structure constants of the gauge group of the reduced theory γ^M_{NP}, m^I_{MN} and β_{MNP}. Obviously, when $\gamma^M_{NP} = 0$ this equation reduces to (3.2), as it should. Note however that the general form of (4.29) must remain completely unchanged regardless of the explicit ansatz for reduction, which contains information about the symmetry algebra. This is a consequence of symmetry. The only effect of a more complicated realization of internal symmetry is to increase the number of independent structure constants, but without changing the form of (4.29).

Hence the consistent reduction Ansätze for our most general massive supergravity which can be derived from any Scherk-Schwarz reduction is given by equations (4.15) and (4.22), plus the appropriate reduction ansatz for the $\mathcal{B}$ field yielding (4.29). Note that we did not have to present the explicit form of the internal coordinate dependence appearing in the basis forms η^M. This dependence is implicitly fixed by the symmetry algebra given in (4.4) and (4.5). In the simplest examples, this dependence may be simply polynomial, but more complicated transcendental functions can also arise. We will complete the reduction of the action to the form which is manifestly invariant under both gauge and U-duality transformations in the next section. As we have discussed in the previous two sections, the gauge symmetries of the reduced theory come in four flavors: the Yang-Mills gauge symmetries which are already present in ten-dimensions; the Kaluza-Klein gauge symmetries which arise from diffeomorphisms on the internal space; the Kalb-Ramond gauge symmetries, which are similarly related to gauge transformations of the two-form on the internal space; and the remaining two-form U(1) gauge symmetry, which however remains decoupled from the one-form gauge symmetries by Lorentz invariance.

5. General massive reductions

At this point, we are ready to carry out the reduction of the action (2.1) from 10 to D dimensions, using the Ansätze (4.15), (4.22) and (4.29), with the set of mass parameters γ^M_{NP}, m^I_{MN} and β_{MNP}. We will not give the detailed computation here, which can be found in the appendix, but merely quote the results. First, it is convenient to split the action into three sectors: the metric-dilaton, Yang-Mills, and Kalb-Ramond three-from, and we discuss each of them separately.

We begin with the metric-dilaton sector, which is given by

$$S_{g\phi} = \int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\Big\{\mathcal{R}(\mathcal{G}) + (\nabla\Phi)^2\Big\}\,. \tag{5.1}$$

Using the ansatz (4.15) and the splitting of the zehnbein in terms of the D-bein e^α and d-bein $\mathcal{E}^A$ defined in (4.21), we can expand the ten-dimensional Ricci scalar and dilaton in terms of fields in the D-dimensional space-time. The action (5.1) becomes

$$\begin{aligned}S_{g\phi} = \int d^Dx\sqrt{-g}e^{-\phi}\Big\{ & R + (\nabla\phi)^2 + \frac{1}{4}\mathcal{D}_\mu\mathcal{G}_{MN}\mathcal{D}^\mu\mathcal{G}^{MN} - \frac{1}{4}\mathcal{G}_{MN}V^M{}_{\mu\nu}V^{N\mu\nu} - \\ & - \mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\gamma^M_{PR}\gamma^N_{QS} - 2\mathcal{G}^{MN}\gamma^P_{MQ}\gamma^Q_{NP}\Big\}\,,\end{aligned} \tag{5.2}$$

where we have used the following definitions:

$$\begin{aligned}\phi &= \Phi - \ln\left(\sqrt{\det(\mathcal{G})}\right) \\ \mathcal{D}_\mu\mathcal{G}_{MN} &= \partial_\mu\mathcal{G}_{MN} - 2\mathcal{G}_{MP}\gamma^P_{NQ}V^Q{}_\mu - 2\mathcal{G}_{NP}\gamma^P_{MQ}V^Q{}_\mu \\ V^M{}_{\mu\nu} &= \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu - 2\gamma^M_{NP}V^N{}_\mu V^P{}_\nu\end{aligned} \tag{5.3}$$

and where the covariant derivative of the moduli $\mathcal{G}_{MN}$ emerges because of the axionic degrees of freedom contained in the matrix $\mathcal{G}_{MN}$: we have $\mathcal{G}_{MN} = \delta_{AB}\mathcal{E}^A{}_M\mathcal{E}^B{}_N$ and the transformation rule (4.19), which induces (4.17). Hence the local derivatives of $\mathcal{G}_{MN}$ must be defined covariantly, since it contains a symmetric bilinear of adjoint fields with charge 1 with respect to the nonabelian Kaluza-Klein group. It can be easily verified that the determinant of $\mathcal{G}_{MN}$ does not contain any axionic fields, however, and so is a gauge singlet. That it why we can still shift the ten-dimensional dilaton in the usual way in (5.3) to get the D-dimensional dilaton.

We can now reduce the Yang-Mills sector. The ten-dimensional action is

$$S_{CYM} = -\frac{1}{4}\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\mathcal{F}^I{}_{\mu\nu}\mathcal{F}^{I\,\mu\nu}\,, \tag{5.4}$$

Using (4.15) and (4.22) as well as the definition $A^I{}_\mu = \mathcal{A}^I{}_\mu - \mathcal{A}^I{}_M V^M{}_\mu$, which is the same as in (2.5), we arrive at

$$\begin{aligned} S_{CYM} = -\int d^Dx\sqrt{-g}e^{-\phi}\Big\{ & \frac{1}{4}(F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu})(F^{I\ \mu\nu} + \mathcal{A}^I{}_M V^{A\ \mu\nu}) + \\ & + \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu \mathcal{A}^I{}_M \mathcal{D}^\mu \mathcal{A}^I{}_N + \\ & + \mathcal{G}^{MP}\mathcal{G}^{NQ}\left(m^I_{MN} + \mathcal{A}^I{}_R\gamma^R_{MN}\right)\left(m^I_{PQ} + \mathcal{A}^I{}_S\gamma^S_{PQ}\right)\Big\} \end{aligned} \tag{5.5}$$

where we use

$$\begin{aligned} \mathcal{D}_\mu \mathcal{A}^I{}_M &= \partial_\mu \mathcal{A}^I{}_M - 2(m^I_{MN} + \mathcal{A}^I{}_P\gamma^P_{MN})V^N{}_\mu \\ F^I{}_{\mu\nu} &= \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu - 2m^I_{MN}V^M{}_\mu V^N{}_\nu\,, \end{aligned} \tag{5.6}$$

which again follow straightforwardly by dimensional reduction. More details can be found in appendix B

The last contribution to the action comes from the three-form kinetic terms in ten dimensions

$$S_{NS} = -\frac{1}{12}\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}\,. \tag{5.7}$$

The reduction of this action, using once again the Ansätze (4.15), (4.22), (4.29) and the field redefinition (2.5), produces the following action in D dimensions:

$$\begin{aligned} S_{NS} = -\int d^Dx\sqrt{-g}e^{-\phi}\Big\{ & \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \\ & +\frac{1}{4}\mathcal{G}^{MN}\Big(H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MP}V^P{}_{\mu\nu}\Big)\Big(H^{\mu\nu}{}_N - \mathcal{A}^I{}_N F^{I\ \mu\nu} - \mathcal{C}_{NQ}V^{Q\mu\nu}\Big) + \\ & +\frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\left(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\right)\left(\mathcal{D}^\mu\mathcal{B}_{PQ} + \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\right) + \\ & +\frac{3}{4}\mathcal{G}^{MQ}\mathcal{G}^{NR}\mathcal{G}^{PS}\left(\beta_{MNP} + 2\mathcal{A}^I{}_{[M}m^I_{NP]} - 2\mathcal{C}_{T[M}\gamma^G_{NP]}\right)\times \\ & \times\left(\beta_{QRS} + 2\mathcal{A}^J{}_{[Q}m^J_{RS]} - 2\mathcal{C}_{U[Q}\gamma^U_{RS]}\right)\Big\} \end{aligned} \tag{5.8}$$

The new definitions here are

$$\begin{aligned} \mathcal{D}_\mu\mathcal{B}_{MN} &= \partial_\mu\mathcal{B}_{MN} + 2m^I_{MN}A^I{}_\mu + 2\gamma^P_{MN}B_{\mu P} - \\ & -\beta_{MNP}V^P{}_\mu + 4\mathcal{B}_{Q[M}\gamma^Q_{N]P}V^P{}_\mu - 2\mathcal{A}^I{}_{[M}m^I_{N]P}V^P{}_\mu \\ H_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP}V^N{}_\mu V^P{}_\nu + \\ & +4\gamma^P_{MN}B_{[\mu P}V^N{}_{\nu]} + 4m^I_{MN}A^I{}_{[\mu}V^N{}_{\nu]} \end{aligned} \tag{5.9}$$

and the reduced three-form field strength is

$$\begin{aligned} H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2}A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{2}V^M{}_\mu H_{\nu\lambda M} - \frac{1}{2}B_{\mu M}V^M{}_{\nu\lambda} + \frac{1}{2}\beta_{MNP}V^M{}_\mu V^N{}_\nu V^P{}_\lambda - \\ -m^I_{MN}A^I{}_\mu V^M{}_\nu V^N{}_\lambda - \gamma^M_{NP}B_{\mu M}V^N{}_\nu V^P{}_\lambda + \text{ cyclic perm.}\,, \end{aligned} \tag{5.10}$$

where $B_{\mu M}$ and $B_{\mu\nu}$ are defined in (2.5), and still are the correct quantities to express the reduced action, in a manifestly gauge and U-duality symmetric way.

To reassemble the reduced terms (5.2), (5.5) and (5.8) into a manifestly symmetric action in D dimensions, we first need to establish the correct gauge algebra of the reduced theory, and identify the gauge invariant couplings of fields. Proceeding as before, we first give the infinitesimal reduced gauge transformations. Details of their derivation can be found in appendix D. As before, the small gauge transformations fall into three categories, listed here in the order of ascending complexity. The fields not listed below explicitly are invariant under the corresponding transformations. The reduced gauge transformations are now:

1. Kalb-Ramond gauge transformations:

$$\begin{aligned} \mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda_P \gamma^P_{MN} \\ B'_{\mu M} &= B_{\mu M} + \partial_\mu \lambda_M - 2\lambda_P \gamma^P_{MN} V^N{}_\mu \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda_M V^M{}_{\mu\nu} + \gamma^M_{NP}\lambda_M V^N{}_\mu V^P{}_\nu\,; \end{aligned} \tag{5.11}$$

2. Yang-Mills gauge transformations:

$$\begin{aligned} \mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda^I m^I_{MN} \\ A'^I{}_\mu &= A^I{}_\mu + \partial_\mu \lambda^I \\ B'_{\mu M} &= B_{\mu M} - 2\lambda^I m^I_{MN} V^N{}_\mu \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda^I F^I{}_{\mu\nu} + m^I_{MN}\lambda^I V^M{}_\mu V^N{}_\nu\,; \end{aligned} \tag{5.12}$$

   (i.e. unchanged from the case of flat torus discussed in section 3), and

3. Kaluza-Klein gauge transformations:

$$\begin{aligned} \mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2\gamma^N_{MP}\omega^P \mathcal{A}^I{}_N + 2m^I_{MN}\omega^N \\ \mathcal{B}'_{MN} &= \mathcal{B}_{MN} + 3\beta_{MNP}\omega^P + 2\mathcal{A}^I{}_{[M} m^I_{N]P}\omega^P + \mathrm{O}(\omega^2) \\ \mathcal{G}'_{MN} &= \mathcal{G}_{MN} + 2\gamma^P_{MQ}\omega^Q \mathcal{G}_{PN} + 2\gamma^P_{NQ}\omega^Q \mathcal{G}_{MP} + \mathrm{O}(\omega^2) \\ V'^M{}_\mu &= V^M{}_\mu - 2\gamma^M_{NP}\omega^P V^N{}_\mu + \partial_\mu \omega^M + \mathrm{O}(\omega^2) \\ A'^I{}_\mu &= A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + \mathrm{O}(\omega^2) \\ B'_{\mu M} &= B_{\mu M} + 2\gamma^N_{MP}\omega^P B_{\mu N} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + \mathrm{O}(\omega^2) \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu - \\ &\quad - 2\gamma^P_{MN}\omega^M B_{[\mu P} V^N{}_{\nu]} - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + \mathrm{O}(\omega^2)\,. \end{aligned} \tag{5.13}$$

Note that in the last set of gauge transformations, we have nontrivial transformation rules for the moduli $\mathcal{G}_{MN}$. This arises from the nontrivial couplings of the metric axions, which were absent in the section 3 where the metric mass parameters γ^M_{NP} were set to zero.

Now as in eq. (3.21), we define a combined gauge parameter

$$\hat{\omega}^a = (\omega^M, \lambda_M, \lambda^I) \tag{5.14}$$

and generators: $T_a = (Z_M, X^M, Y^I)$. The algebra of the latter

$$[T_a, T_b] = i f_{ab}{}^c T_c \tag{5.15}$$

defines the new set of structure constants, $f_{ab}{}^c$. To compute these, we again consider the products of transformations (5.11)–(5.13) of the form $h^{-1} \cdot g^{-1} \cdot h \cdot g$ where h and g are two of gauge transformations with $g = \exp(i\hat{\omega}_1^a T_a)$ and $h = \exp(i\hat{\omega}_2^a T_a)$. Hence substituting the explicit form of the gauge transformations (5.11)–(5.13), we can deduce the structure constants. We evaluate $h^{-1} \cdot g^{-1} \cdot h \cdot g|\Psi\rangle$ for the set of basis states defined by the vector fields. The structure constants are, as expected,

$$f^M{}_{NP} = f_{NP}{}^M = 2\gamma^M_{NP}\,, \qquad f^I{}_{MN} = f_{MN}{}^I = 2m^I_{MN}\,, \qquad f_{MNP} = -3\beta_{MNP}\,. \tag{5.16}$$

The resulting gauge algebra, which encompasses the case given in (3.25), is

$$\begin{aligned}
[X^M, X^N] &= [Y^I, Y^J] = [X^M, Y^I] = 0 \\
[X^M, Z_N] &= 2i\gamma^M_{NP} X^P \\
[Y^I, Z_M] &= 2im^I_{MN} X^N \\
[Z_M, Z_N] &= -3i\beta_{MNP} X^P + 2im^I_{MN} Y^I + 2i\gamma^P_{MN} Z_P\,.
\end{aligned} \tag{5.17}$$

While the standard Cartan metric on this Lie algebra (5.17) is still degenerate, since the gauge algebra is not semi-simple, we can nevertheless define the metric on the gauge algebra by $\langle T_a, T_b\rangle = L_{ab}$, exactly as before.

Formally keeping the definition of the Lie-algebra-valued gauge field one-form potential given in eq. (3.26), we find that the Lie-algebra-valued gauge field strength

$$F = dA + iA \wedge A = \frac{1}{2} F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu \tag{5.18}$$

again has components which coincide with the expressions for field strengths that come from dimensional reduction: $F^a_{\mu\nu} = (V^M{}_{\mu\nu}, H_{\mu\nu M}, F^I_{\mu\nu})$. Explicitly, the components of the gauge field strength are

$$\begin{aligned}
F_{\mu\nu}{}^M &= \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu - 2\gamma^M_{NP} V^N{}_\mu V^P{}_\nu \\
F_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP} V^N{}_\mu V^P{}_\nu + 4m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + 4\gamma^P_{MN} B_{[\mu P} V^N{}_{\nu]} \\
F_{\mu\nu}{}^I &= \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu - 2m^I_{MN} V^M{}_\mu V^N{}_\nu\,.
\end{aligned} \tag{5.19}$$

The nonabelian Chern-Simons form can be computed as usual, to yield

$$\begin{aligned}
\omega_{CS} &= \langle A \wedge F - \frac{i}{3} A \wedge A \wedge A\rangle \qquad (5.20) \\
&= \frac{1}{3} dx^\mu \wedge dx^\nu \wedge dx^\lambda \Big(\frac{1}{2} A^I{}_\mu F^I{}_{\nu\lambda} + \frac{1}{2} V^M{}_\mu H_{\nu\lambda M} + \frac{1}{2} B_{\mu M} V^M{}_{\nu\lambda} + \\
&\quad + \frac{1}{2}\beta_{MNP} V^M{}_\mu V^N{}_\nu V^P{}_\lambda - m^I_{MN} A^I{}_\mu V^M{}_\nu V^N{}_\lambda - \gamma^M_{NP} B_{\mu M} V^N{}_\nu V^P{}_\nu + \text{c.p.} \Big)\,.
\end{aligned}$$

This is exactly the anomaly contribution to the reduced three-form field strength. Again the Chern-Simons terms get twisted together into a single nonabelian structure, such that, in form notation, the reduced three-form field strength is simply

$$H = dB - \frac{1}{2}\omega_{CS} \, . \tag{5.21}$$

We can also put the moduli potential in a duality symmetric form. Lowering the last index on the structure constants, $f_{abc} = f_{[ab|}{}^{d}L_{d|c]}$, we can write the scalar potential entirely in terms of the moduli matrix M^{ab}:

$$W(M) = \frac{1}{12}M^{ad}M^{be}M^{cf}f_{abc}f_{def} - \frac{1}{4}M^{ad}L^{be}L^{cf}f_{abc}f_{def} \, . \tag{5.22}$$

Note the additional term linear in M which was absent in the formula (3.32) obtained with nontrivial fluxes in the matter sector. This is, of course, consistent with our previous results. Using the structure constants found in eq. (3.24), i.e. $\gamma^M_{NP} = 0$, this linear term automatically vanishes. This new term's only nontrivial contribution here is the last interaction appearing in eq. (5.2), which is linear in $\mathcal{G}$ and quadratic in the γ^M_{NP}.

The mass parameters γ^M_{NP}, m^I_{MN} and β_{MNP} must satisfy the constraints

$$\begin{aligned} \gamma^M_{N[P}\gamma^N_{QR]} &= 0 \\ \gamma^M_{MN} &= 0 \\ m^I_{Q[M}\gamma^Q_{NP]} &= 0 \\ 3\beta_{R[MN}\gamma^R_{PQ]} &= m^I_{[MN}m^I_{PQ]} \, . \end{aligned} \tag{5.23}$$

These relations will ensure that the structure constants satisfy the Jacobi identity:

$$L^{ad}f_{a[bc|}f_{d|ef]} = \frac{1}{3}L^{ad}\Big(f_{abc}f_{def} + f_{abe}f_{dfc} + f_{abf}f_{dce}\Big) = 0 \, . \tag{5.24}$$

It now remains to note that we can also collect the gauge kinetic terms and the moduli terms in the manifestly gauge- and duality-invariant fashion. The gauge fields transform according to

$$F'^a_{\mu\nu} = U^a{}_b F^b_{\mu\nu} \, , \tag{5.25}$$

where $U^a{}_b$ is an $\mathrm{O}(d, d+16)$ matrix. Thus the moduli fields transform according to

$$M'^{ab} = U^a{}_c U^b{}_d M^{cd} \, , \tag{5.26}$$

under the same gauge transformations, and we see that the covariant derivative of the scalar moduli can be written as

$$\mathcal{D}_\mu M^{ab} = \partial_\mu M^{ab} - f_{cd}{}^{a}A^c_\mu M^{db} - f_{cd}{}^{b}A^c_\mu M^{ad} \, . \tag{5.27}$$

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These are indeed identical with the expressions which were obtained by reduction. With this it can be easily shown that the gauge kinetic terms can be collected into the covariant expression

$$\mathcal{F}^2 = F^a_{\mu\nu} L_{ab} M^{bc} L_{cd} F^{d\mu\nu} \,. \tag{5.28}$$

Therefore, the reduced action can be again rewritten as

$$S = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 + \frac{1}{8} L_{ab} \mathcal{D}_\mu M^{bc} L_{cd} \mathcal{D}^\mu M^{da} - \\ - \frac{1}{4} F^a_{\mu\nu} L_{ab} M^{bc} L_{cd} F^{d\mu\nu} - \frac{1}{12} H^2_{\mu\nu\lambda} - W(M) \Big\} \,, \tag{5.29}$$

which is formally the same as (3.38), albeit the structure of the gauge group is more complicated. Hence as before, the general massive reductions produce reduced theories with a remarkably symmetric form, where a part of the $\mathrm{O}(d, d+16)$ duality must be gauged in order to accommodate the couplings induced by the mass terms.

6. Discussion

Above we have shown that, under generalized Scherk-Schwarz reductions on a twisted d-dimensional torus, low energy heterotic string theory is manifestly invariant under $\mathrm{O}(d, d+16)$. The essential step in the construction of the reduced action is to realize that the mass parameters can be assembled into a completely antisymmetric three-index tensor f_{abc} with $\mathrm{O}(d, d+16)$ indices. These parameters serve two roles in the reduced theory: as the mass parameters defining the scalar potential (5.22), and also as the structure constants in the *non-abelian* gauge group. Thus the reduced theory is now a massive gauged supergravity. By inspecting the reduced action (5.29), we discover that usual U-duality transformations, $M \to M' = \Omega M \Omega^T$, $F \to F' = \Omega F$, are now accompanied by

$$f_{abc} \to f'_{abc} = \Omega_a{}^d \Omega_b{}^e \Omega_c{}^f f_{def} \,. \tag{6.1}$$

Given a certain set of mass parameters, U-duality transformations will in general map these to a new collection of modified couplings. In so doing, U-duality maps one *reduced* theory to another. Thus we can think that while U-duality remains a symmetry of the full theory, it is spontaneously broken in any given compactification. Our U-duality covariant formalism provides a unified framework in which to investigate the "disparate" massive supergravities, appearing in generalized reductions of heterotic string theory.

It is interesting to consider the symmetries identified here in the framework of ref. [28]. There, proper symmetries are identified as those that act on fields, while pseudo-symmetries act on both fields and coupling constants. In the present case, from the point of view of the ten-dimensional heterotic string theory (or its low energy supergravity limit), the $\mathrm{O}(d, d+16)$ transformations are proper symmetries. The key point is that the mass parameters f_{abc} in eq. (6.1) were identified with nontrivial background fields in the internal space. However having compactified, one could truncate the reduced theory to the D-dimensional gauged supergravity for which eq. (5.29) is the bosonic part of the action. From the point of view of this truncated theory, the $\mathrm{O}(d, d+16)$ duality group contains both proper symmetries and pseudo-symmetries since the f_{abc} are now simply coupling constants. The proper symetries would be the subgroup of transformations leaving a given set of couplings invariant under the above action (6.1), while the pseudo-symmetries act nontrivially. It is in this sense that we mean that in general U-duality maps one *reduced* theory to another.

A novel feature in the present case is that the background fields that produce this spontaneous breaking are topological. That is the corresponding fluxes or geometric curvatures on the internal space are not associated with dynamical fields in the theory. One aspect of the mass parameters, which we have not considered above, is the fact that they must be quantized. The usual Dirac quantization conditions arise for the Yang-Mills fluxes from the consideration of charged string states moving in such a background. Similarly, the three-form fluxes are shown to be quantized by considering strings with nontrivial winding numbers [32]. Finally for compactification on a twisted torus, properly matching the geometry of the d-torus will require that the metric twists lie in $\mathrm{SL}(d, \mathbb{Z})$. Given the quantized nature of the mass parameters, it would seem that the U-duality symmetry is broken down to $\mathrm{O}(d, d+16, \mathbb{Z})$. It is natural to conjecture then that these discrete symmetries are in fact an exact symmetry of the full string theory, just as in the case of the standard toroidal compactifications [29].

Here, we are lead to a slight puzzle. Quantization conditions seem to break the $\mathrm{O}(d, d+16, \mathbb{R})$ symmetry of the reduced action (5.29) to $\mathrm{O}(d, d+16, \mathbb{Z})$. Another aspect of the reduced theory was that a part of the global $\mathrm{O}(d, d+16, \mathbb{R})$ symmetry becomes a local gauge symmetry. Thus it would seem that this continuous subgroup of the U-duality group must also be exact since it corresponds to constant gauge transformations. The puzzle is to understand the interplay of these two apparent exact symmetries. We will argue that in fact these symmetries are distinct symmetries, despite their apparent common origin in $\mathrm{O}(d, d+16, \mathbb{R})$.

This distinction is most easily understood by examining a concrete example. The simplest case to consider is a generalized reduction on T^3 with a metric twist. As described in section 4, one can think of the resulting geometry as an $\mathrm{SL}(2, \mathbb{Z})$ fibre bundle with a T^2 fibre over an S^1 base. That is as one moves around the S^1, the T^2 geometry is varying by a real shift of the modular parameter, $\tau \to \tau + a$. Having circumnavigated the circle once, in the simplest case, the total shift is $a = 1$, allowing the T^2 fibres to be identified by an $\mathrm{SL}(2, \mathbb{Z})$ transformation. Now we note that in the reduced theory an arbitrary shift of the metric axion, i.e. the T^2 modulus τ, is a symmetry. In the ten-dimensional space, while this shift changes the T^2 geometry locally over a fixed point on the S^1, the geometry is in fact invariant if the axion shift is accompanied by a translation in the S^1 direction. That is all possible T^2 geometries can be found in the fibre over some point in the S^1, hence a shift of the T^2 modulus along with a corresponding translation along the S^1 must be a symmetry. Another point of view would be that in a standard reduction, a constant translation along the S^1 corresponds to a constant U(1) symmetry transformation, which acts trivially in the reduced theory. In the generalized reduction considered here, these translations alone are no longer a symmetry because of the twisted geometry. Instead, to produce a symmetry the translation is accompanied by a shift of the metric axion. In any event, the global gauge transformation which produces $\tau \to \tau + 1$ is accompanied by a translation once around the S^1. The U-duality symmetry producing the same shift of τ makes no translation in the S^1 direction. Hence these two $\mathrm{O}(d, d + 16)$ transformations are actually distinct ten-dimensional symmetries.

The nontrivial interplay of the gauge and U-duality symmetries seems to modify the form of the moduli space of these theories. For a standard toroidal compactification of the heterotic string, the moduli space[26] is $\mathrm{SO}(d, d + 16, \mathbb{Z})\backslash\mathrm{SO}(d, d + 16, \mathbb{R})/(\mathrm{SO}(d, \mathbb{R}) \times \mathrm{SO}(d + 16, \mathbb{R}))$. For the generalized axion compactifications considered here, it appears that the moduli space is reduced to $(H \times \mathrm{SO}(d, d + 16, \mathbb{Z}))\backslash\mathrm{SO}(d, d+16, \mathbb{R})/(\mathrm{SO}(d, \mathbb{R}) \times \mathrm{SO}(d+16, \mathbb{R}))$ where H is the group of global gauge transformations. One might have included the global gauge transforms for the standard compactification, however, their action is trivial in that case and so they have no effect. Note that the additional reduction of modding out by H affects even the local structure of the moduli space.

Here we use the term "moduli space" loosely, as typically the scalar potential (5.22) pushes the scalar moduli to evolve under its influence. For example, one might choose to set $M^{ab} = \delta^{ab}$ as initial data, but this configuration typically would not extremize the potential and thus the scalars would begin to vary. One might observe that naively a simple solution for moduli that always extremizes the potential is $M^{ab} \propto L^{ab}$. Unfortunately, this is not a physical solution as it corresponds to a vanishing internal metric. Instead the natural vacua of such theories are therefore similar to the linear dilaton vacua [30]. The natural supersymmetric vacua are typically domain walls in the reduced theory [1, 2]. One is also led to consider cosmological evolutions in such theories [7].

Although it should be obvious, we also mention that the appearance of a non-abelian gauge symmetry in the generalized reductions is very different than the symmetry enhancement that arises at special points in the moduli space of the standard toroidal reduction. In the latter case, new massless degrees of freedom associated with string winding modes appear at these special points to enlarge the gauge group. In the generalized reductions, there are no extra massless fields appearing in the reduced theory. Actually quite the opposite effect should take place. If one examines the detailed form of the covariant derivatives of the moduli in eqs. (5.6) and (5.9), one recognizes that these axion scalars are actually Stuckelberg fields. The same is true in eq. (5.3) for the metric axions if one considers fluctuations around a fixed background metric. The Stuckelberg nature of the axions is also evident from the gauge transformations in eqs. (5.11)–(5.13). Of course, this should have been anticipated given that it was the axionic shift symmetry which was gauged in the generalized reduction. In any event, the associated gauge fields will in fact become massive, and the original axions are traded for massive longitudinal modes. So in contrast with the usual enhanced symmetry points where the number of massless modes increases, the generalized reductions actually produce a reduction in the number of (strictly) massless degrees of freedom as compared to the standard compactification.

It should be mentioned, however, that in ref. [20] a unified framework was constructed to incorporate all of the enhanced symmetry points of the standard toroidal reduction to four dimensions. The form of the "completely duality-invariant low-energy effective action" constructed there is in fact very similar to our action in eq. (5.29). The similarity of these two actions for four dimensions is essentially dictated by the fact that they are both versions of gauged $N = 4$ supergravity in which the form of the action is completely fixed given the gauge group [21]. Undoubtedly, it is also true that supersymmetry essentially fixes the elegant form of our bosonic action (5.29) in higher dimensions, as well. In particular, the simple form of the moduli potential (5.22) is probably a result of the large number of supersymmetries in the theory.

Given the above discussion, a natural question is under what conditions do we expect the action (5.29) to give a reasonable description of the low energy degrees of freedom. In our derivation, we are ignoring[3] massive string states with masses $m_s \simeq 1/\sqrt{\alpha'} = 1/\ell_s$, Kaluza-Klein momentum modes with $m_{KK} \simeq 1/L$ and string winding modes with $m_w \simeq L/\ell_s^2$, where L is the typical size of the compact dimensions. Hence we should require that any of the masses appearing in our generalized Scherk-Schwarz reductions are smaller than these. Otherwise, the corresponding massive states should also be integrated out, or alternatively we could consider the full string theory including all of the massive states. Given the flux and curvature quantization conditions discussed above, one must be careful in considering this question.

Considering each type of the mass contribution individually, we begin with the three-form fluxes. In this case [32], the quantization condition takes the form: $\int_{3-cycle} H \simeq n\,\ell_s^2$ where n is an integer. Therefore one finds that the corresponding mass parameters must be $\beta_{MNP} \simeq \ell_s^2/L^3$. Thus if one is considering a regime where $\ell_s/L \ll 1$, i.e. the internal space is large compared to the string scale, then these masses are much smaller than those of the massive states listed above. Simi-

[3]More accurately, we should think that these states are integrated out in constructing the effective theory.

larly, considering the gauge field fluxes, we have $\int_{2-cycle} F^I \simeq n\,\ell_s$ and $m^I_{MN} \simeq \ell_s/L^2$. Thus in the regime $\ell_s/L \ll 1$, these gauge flux masses are smaller than those of the states which have been ignored, however, they are systematically larger than those generated by the three-form fluxes. The latter though assumes that all of the compact dimensions are essentially the same size.

Finally we consider the geometric masses for compactification on a twisted torus. In this case, it is easiest to gain insight by considering the case of T^3 for which some of the explicit formulae were presented at the beginning of section 4. It is straightforward to see that in general $\gamma^M_{NP} \simeq L_M/(L_N L_P)$ where L_M is the compactification size associated with y^M. Therefore if all of the compact dimensions are of the same size, these masses will be of the same order as that of the Kaluza-Klein modes, and so one is not really justified in using the action (5.29) as it stands. By arranging a less democratic compactification, however, in which $L_M/L_N \ll 1$ for certain directions, one can safely introduce certain γ^M_{NP} while keeping only the degrees of freedom appearing in (5.29).

The preceding analysis assumed that the sizes of the compact directions were fixed. However, as discussed above, this will likely not be the case due to the scalar potential (5.22). Hence in examining solutions of the equations of motion, one should be aware that the scalars may evolve to (or from) a regime where the low energy theory described by (5.29) is no longer valid.

We have already noted that in section 4 the explicit form of the internal coordinate dependence appearing in the basis forms η^M is not explicitly needed. Rather, this dependence is implicitly fixed by the internal symmetry algebra given in (4.4) and (4.5). The key for inducing the metric mass parameters was the presence of a nonabelian isometry group on the internal manifold. Thus our formalism includes more general nonabelian Kaluza-Klein reductions [5, 33, 34] than the case of twisted tori. In general a reduction must satisfy the consistency condition that the procedures of dimensional reduction and variation of the action commute. Further one usually wants to construct a reduced theory with a finite number of field degrees of freedom, which are generally in one-to-one correspondence with the degrees of freedom of the abelian reductions on internal tori. As it has been discussed in particular in [34], these two conditions are in fact related. A sufficient condition for finite-dimensional truncations is that the pullbacks of generators of the isometry algebra on the tangent space span a complete, linearly independent, orthogonal basis. The end result is then that the permissible internal manifolds should be homogeneous spaces with isometry algebra of dimension equal to the dimension of the manifold on which it acts.

In lower dimensions, the relevant isometry algebras can be completely classified [35]. In two dimensions, the only admissible Lie algebras with two generators are the abelian one and an algebra defined by $[Z_1, Z_2] = QZ_2$, however the only structure constant of the latter is $\gamma^2_{12} = Q$, which violates eq. (4.7) when $Q \neq 0$.

In three-dimensions, the algebras of interest are given by five of the nine Bianchi model geometries [35]. They are Bianchi I (abelian), the two algebras corresponding to the twisted tori: Bianchi II (discussed in [19] and in section (4) and Bianchi VIII (a twisted torus with a cubic ansatz in internal coordinates) and two non-toroidal algebras: a special Bianchi VI with the $h = -1$ (which must be chosen to satisfy eq. (4.7)), and Bianchi IX (corresponding to a group manifold of $\mathrm{SO}(3) \approx \mathrm{SU}(2)/Z_2$ or its universal covering space $\mathrm{SU}(2) \approx S^3$, and with the SU(2) algebra). The latter two algebras give rise to the reduced theories of the same form as (5.29). The 3-sphere with antipodal identification $\mathrm{SU}(2)/Z_2$ has nontrivial 1-cycles, and Wilson loops could still be present, breaking the vector supermultiplet gauge group to $\mathrm{U}(1)^{16}$. Hence it appears to be covered by our consideration, at least in the supergravity limit. In fact reductions of the $N = 1 D = 10$ supergravity on $S^3 \times S^3$ have been considered [36], and in this case because of the absence of the vector supermultiplet, the global topology on the sphere did not play a role. Nevertheless the complete understanding of the case of S^3, and higher-dimensional simply connected manifolds in heterotic string theory would require a more detailed scrutiny of the gauge sector. In addition, since the internal subalgebra is semisimple, the sizes of the internal radii would have to be equal, giving all the geometric masses γ^M_{NP} of the order of $1/L$. Thus as indicated in the discussion above, we can not use the action (5.29) as it stands to consistently study the dynamics of the low energy theory. One must either extend it to include other massive Kaluza-Klein states, which have been integrated out in our construction, or exclude the modes appearing in our action with masses of the order of $1/L$. In any event, this case therefore requires further scrutiny.

In closing, we underline that the Scherk-Schwarz reduction [5] may be viewed as an application of the nonabelian Kaluza-Klein reduction [33, 34] on a group which admits finite-dimensional truncations to supergravity theories. As discussed by Scherk and Schwarz in their work [5], their main motivation was seeking ways to break supersymmetry. Actually, this approach to supersymmetry breaking is really similar to supersymmetry breaking in various p-brane configurations, with the only additional requirement that the p-brane solutions admit a simply transitive isometry group in spatial directions. Such solutions can be interpreted as spontaneous compactification, such as the Freund-Rubin solutions [27] (which happen not to break any supersymmetry, since they correspond to spontaneous compactifications to $AdS_4 \times S^7$ spacetime, but their extensions [37] may do so). Note that the original terminology of Scherk and Schwarz [5] distinguished between the so-called internal and external axionic symmetries. The former were associated with the axions that emerge from the metric of the internal manifold, while the latter come from the form-field sector. However, in light of our interpretation of the $\mathrm{O}(d, d + 16)$ dualities as the maps between the internal and external masses, it is clear that this distinction between the internal and external symmetries is somewhat artificial in string theory.

Moreover, the physically meaningful properties of the reduced theory, given by the masses and the structure constants, depend on the directions and types of fields which are excited on the internal space. In general, the internal fields can be turned on by using the tensor representations of isometries. In this context, a very natural question to ask is what is the most general set of admissible structure constants f_{abc} which arise from the Scherk-Schwarz reduction? It is obvious from our discussion that the answer is equivalent to classifying all inequivalent isometry groups which may act on a compact manifold of dimension d. In other words, this means that to determine all massive supergravities which come from the Scherk-Schwarz dimensional reduction, one should find all inequivalent internal axionic groups, and using them deduce the external groups, which satisfy the Jacobi consistency conditions. The mass terms are related by duality symmetry. If the reductions are performed over group spaces without nontrivial 1-cycles, or if the Wilson loops in the Yang-Mills sector vanish in their own right, the original Yang-Mills gauge symmetry remains unbroken. Hence, the full nonabelian structure of the vector supermultiplet fields should be considered. It may be interesting to consider the relationship of duality and Lie algebras of the Yang-Mills sector and the internal manifold in this case. One may further ask if there may be massive supergravities which cannot be obtained by Scherk-Schwarz reductions, but could be dual to them, in a manner which we have discussed here. Other interesting questions which may be asked in this context are what happens in type I and II superstring theories. These tasks are beyond the scope of the present article, but clearly merit further investigation. It therefore appears appropriate to end our discussion here on this note, and leave these questions for future work.

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A. Reduction of curvature and dilaton

Here we give the main formulas for the dimensional reduction of the metric-dilaton sector of the action (2.1). The metric-dilaton part is given in eq. (5.1), which we repeat here:

$$S_{g\phi} = \int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\Big\{\mathcal{R}(\mathcal{G}) + (\nabla\Phi)^2\Big\}. \qquad \text{(A.1)}$$

The reduction ansatz for the metric is as in eq. (4.15)

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \mathcal{G}_{MN}(x)\left(\eta^M(y) + V^M{}_\mu(x)dx^\mu\right)\left(\eta^N(y) + V^N{}_\nu(x)dx^\nu\right), \quad \text{(A.2)}$$

wherefore we see that we can define the torsion-free tangent space basis according to $\hat{e}^a = \{e^\alpha, \mathcal{E}^A\}$, where

$$\begin{aligned} e^\alpha &= e^\alpha{}_\mu dx^\mu \\ \mathcal{E}^A &= \mathcal{E}^A{}_N(\eta^N + V^N{}_\mu dx^\mu)\,. \end{aligned} \quad \text{(A.3)}$$

Using (A.3), the torsion-free condition $d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b = 0$, and the metric compatibility condition $\hat{\omega}_{ab} = -\hat{\omega}_{ba}$, where the indices are raised and lowered with the flat-space metric $\hat{\eta}_{ab} = (\eta_{\alpha\beta}, \delta_{AB})$, we can determine the spin connexion. Recalling that the non-zero structure constants imply $d\eta^M = -\gamma^M_{NP}\eta^N \wedge \eta^P$ (4.5), after straightforward but tedious algebra we find

$$\begin{aligned} \hat{\omega}^\alpha{}_\beta &= \omega^\alpha{}_\beta - \frac{1}{2}\mathcal{E}_{MN}V^N{}_{\mu\nu}e^{\alpha\mu}e_\beta{}^\mu\mathcal{E}^M \\ \hat{\omega}^A{}_\alpha &= \frac{1}{2}\mathcal{E}^A{}_N V^N{}_{\mu\nu}e_\alpha{}^\mu dx^\nu + \frac{1}{2}\left(\mathcal{E}^{AM}\mathcal{D}_\mu\mathcal{E}_{BM} + \mathcal{E}_B{}^M\mathcal{D}_\mu\mathcal{E}^A{}_M\right)e_\alpha{}^\mu\mathcal{E}^B \\ \hat{\omega}^A{}_B &= \left(\gamma^M_{NP}\mathcal{E}_{CM}\mathcal{E}^{AN}\mathcal{E}_B{}^P + \gamma^M_{NP}\mathcal{E}_{BM}\mathcal{E}^{AN}\mathcal{E}_C{}^P - \gamma^M_{NP}\mathcal{E}^A{}_M\mathcal{E}_B{}^N\mathcal{E}_C{}^P\right)\mathcal{E}^C - \\ &\quad -\frac{1}{2}\left(\mathcal{E}_B{}^M\mathcal{D}_\mu\mathcal{E}^A{}_M - \mathcal{E}^{AM}\mathcal{D}_\mu\mathcal{E}_{BM}\right)dx^\mu\,. \end{aligned} \quad \text{(A.4)}$$

Here $\omega^\alpha{}_\beta$ is the spin connexion of the reduced metric $g_{\mu\nu}$. The covariant derivative $\mathcal{D}_\mu$ is defined according to

$$\mathcal{D}_\mu\mathcal{E}^A{}_N = \partial_\mu\mathcal{E}^A{}_N - 2\mathcal{E}^A{}_M\gamma^M_{NP}V^P{}_\mu \quad \text{(A.5)}$$

and we see that the first index on $\mathcal{E}^A{}_N$ is counting the number of independent objects $\mathcal{E}$, while the second counts the components of each object and hence carries gauge charge with respect to $V^M{}_\mu$. Comparing this with the transformation rule for $\mathcal{E}^A{}_N$ given in (4.19) and the transformation rule for the gauge field $V^M{}_\mu$ (4.16), we find that

$$\mathcal{D}'_\mu\mathcal{E}'^A{}_N = \mathcal{S}_N{}^M\mathcal{D}_\mu\mathcal{E}^A{}_M\,. \quad \text{(A.6)}$$

Hence the ansatz for the reduction of the metric (A.2) ensures that the reduced quantities are automatically gauge-covariant, as intended. Next, we can use the form-notation for the Einstein-Hilbert Lagrangian to simplify the algebra of the reduction of the action. We have in general

$$\mathcal{L}_{EH} = \frac{1}{(D+d-2)!}\epsilon_{a_1\ldots a_{D+d}}\mathcal{R}^{a_1a_2}\wedge\hat{e}^{a_3}\wedge\ldots\wedge\hat{e}^{a_{D+d}} = \sqrt{-\hat{\mathcal{G}}}\mathcal{R}d^Dxd^dy\,. \quad \text{(A.7)}$$

The curvature forms are defined in terms of the spin connexion as $\mathcal{R}_{a_1 a_b} = d\hat{\omega}_{a_1 a_2} + \hat{\omega}_{a_1 b} \wedge \hat{\omega}^b{}_{a_2}$. If we now look at the specific contractions in (A.7), we see that we can write

$$\mathcal{L}_{EH} = \mathcal{J}_1 + 2\mathcal{J}_2 + \mathcal{J}_3\,, \tag{A.8}$$

where

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{(D+d-2)!}\epsilon_{\alpha\beta a_1 \ldots a_{D+d-2}} \mathcal{R}^{\alpha\beta} \wedge \hat{e}^{a_1} \wedge \ldots \wedge \hat{e}^{a_{D+d-2}} \\
\mathcal{J}_2 &= \frac{1}{(D+d-2)!}\epsilon_{\alpha A a_1 \ldots a_{D+d-2}} \mathcal{R}^{\alpha A} \wedge \hat{e}^{a_1} \wedge \ldots \wedge \hat{e}^{a_{D+d-2}} \\
\mathcal{J}_3 &= \frac{1}{(D+d-2)!}\epsilon_{AB a_1 \ldots a_{D+d-2}} \mathcal{R}^{AB} \wedge \hat{e}^{a_1} \wedge \ldots \wedge \hat{e}^{a_{D+d-2}}\,.
\end{aligned} \tag{A.9}$$

Now, these formulas give useful calculational shortcuts. By the structure of the indices in the ϵ-tensor of the first expression, one can readily see that the factor $\hat{e}^{a_1} \wedge \ldots \wedge \hat{e}^{a_{D+d-2}}$ must contain all d-beins $\mathcal{E}$. As a result, only those terms in $\mathcal{R}^{\alpha\beta}$ which are independent of $\mathcal{E}^A$ contribute to $\mathcal{J}_1$. Since

$$\mathcal{R}^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{4}\mathcal{G}_{MN} V^M{}_{\mu\nu} V^N{}_{\lambda\sigma} e^{\alpha\mu}(e^{\beta\nu} dx^\lambda + e^{\beta\lambda} dx^\nu) \wedge dx^\sigma + \mathrm{O}(\mathcal{E}^M)\,, \tag{A.10}$$

where $V^M{}_{\mu\nu} = \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu - 2\gamma^M_{NP} V^N{}_\mu V^P{}_\nu$ is precisely the correct nonabelian field strength as defined in (5.3). Factoring out the invariant measure on the internal space, we have $\hat{V}_{10} = e^1 \wedge \ldots \wedge e^{D+N} = \sqrt{-g}\sqrt{\mathcal{G}} d^D x d^d y$. With this, we finally find

$$\mathcal{J}_1 = \sqrt{-g}\sqrt{\mathcal{G}} d^D x d^d y \left(R - \frac{3}{4}\mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\mu\nu} \right). \tag{A.11}$$

Similar calculations lead to the expressions for $\mathcal{J}_2$ and $\mathcal{J}_3$. Noting that it is convenient to define

$$\mathcal{D}_\mu \mathcal{G}_{MN} = \partial_\mu \mathcal{G}_{MN} - 2\mathcal{G}_{MP}\gamma^P_{NQ} V^Q{}_\mu - 2\mathcal{G}_{PN}\gamma^P_{MQ} V^Q{}_\mu\,, \tag{A.12}$$

which comes directly from (A.6) and the fact that since $\mathcal{E}^A{}_N$ are d objects which transform in the adjoint representation of the reduced gauge group, the matrix $\mathcal{G}_{MN}$ transforms in the symmetric direct product of two adjoints. With this, we can show that

$$\begin{aligned}
\mathcal{J}_3 = \sqrt{-g}\sqrt{\mathcal{G}} d^D x d^d y \Bigg(&\frac{1}{4}\mathcal{G}^{MN}\mathcal{G}^{PQ}\mathcal{D}_\mu \mathcal{G}_{MP} \mathcal{D}^\mu \mathcal{G}_{NQ} - \frac{1}{4}\mathcal{G}^{MN}\mathcal{D}_\mu \mathcal{G}_{MN} \mathcal{G}^{PQ}\mathcal{D}_\mu \mathcal{G}_{PQ} - \\
&- \mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\gamma^M_{PR}\gamma^N_{QS} - 2\mathcal{G}^{MN}\gamma^P_{MQ}\gamma^Q_{NP} \Bigg)
\end{aligned} \tag{A.13}$$

and after some more involved algebra, and the definition of the covariant derivative of a gauge-charged reduced base space tensor according to

$$\begin{aligned}
\mathcal{D}_\mu \Phi^{M_1 \ldots M_k\ \nu_1 \ldots \nu_p} = {}& \partial_\mu \Phi^{M_1 \ldots M_k\ \nu_1 \ldots \nu_p} + \Gamma^{\nu_1}_{\mu\rho}\Phi^{M_1 \ldots M_k\ \rho \ldots \nu_p} + \cdots + \Gamma^{\nu_p}_{\mu\rho}\Phi^{M_1 \ldots M_k\ \nu_1 \ldots \rho} + \\
& + 2q\gamma^{M_1}_{NP} V^P{}_\mu \Phi^{N \ldots M_k\ \nu_1 \ldots \nu_p} + \cdots + 2q\gamma^{M_k}_{NP} V^P{}_\mu \Phi^{M_1 \ldots N\ \nu_1 \ldots \nu_p},
\end{aligned} \tag{A.14}$$

where q is the unit of charge of $\Phi^{M_1 \ldots M_k\ \nu_1 \ldots \nu_p}$, we find that

$$\mathcal{J}_2 = \sqrt{-g}\sqrt{\mathcal{G}} d^D x d^d y \left(\frac{1}{4} \mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\mu\nu} + \frac{1}{4} \mathcal{D}_\mu \mathcal{G}_{MN} \mathcal{D}^\mu \mathcal{G}^{MN} - \frac{1}{2} \mathcal{D}_\mu (\mathcal{G}^{MN} \mathcal{D}^\mu \mathcal{G}_{MN}) \right) . \tag{A.15}$$

Hence by combining (A.11), (A.13) and (A.15), we can write down the metric-dilaton action (A.1) as

$$\begin{aligned} S_{g\phi} = \int d^D x \sqrt{-g}\sqrt{\mathcal{G}} e^{-\Phi} \Big(R &- \frac{1}{4} \mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\mu\nu} - \frac{1}{4} \mathcal{G}^{MN} \mathcal{D}_\mu \mathcal{G}_{MN} \mathcal{G}^{PQ} \mathcal{D}^\mu \mathcal{G}_{PQ} + \\ &+ \frac{1}{4} \mathcal{D}^\mu \mathcal{G}^{MN} \mathcal{D}_\mu \mathcal{G}_{MN} - \frac{1}{2} \mathcal{D}_\mu (\mathcal{G}^{MN} \mathcal{D}_\mu \mathcal{G}_{MN}) - \\ &- \mathcal{G}_{MN} \mathcal{G}^{PQ} \mathcal{G}^{RS} \gamma^M_{PR} \gamma^N_{QS} - 2\mathcal{G}^{MN} \gamma^P_{MQ} \gamma^Q_{NP} + (\nabla\Phi)^2 \Big) , \end{aligned} \tag{A.16}$$

where we have canceled the constant factor $\int d^d y$. Defining the reduced dilaton field as in (2.4), $\exp(-\phi) = \sqrt{\mathcal{G}} \exp(-\Phi)$, we can integrate terms in (A.16) by parts, and dropping the boundary terms (which is in fact required by duality as discussed in [22]), we find the final answer

$$\begin{aligned} S_{g\phi} = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 &+ \frac{1}{4} \mathcal{D}_\mu \mathcal{G}_{MN} \mathcal{D}^\mu \mathcal{G}^{MN} - \frac{1}{4} \mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\mu\nu} - \\ &- \mathcal{G}_{MN} \mathcal{G}^{PQ} \mathcal{G}^{RS} \gamma^M_{PR} \gamma^N_{QS} - 2\mathcal{G}^{MN} \gamma^P_{MQ} \gamma^Q_{NP} \Big\} . \end{aligned} \tag{A.17}$$

This completes the reduction of the metric-dilaton action.

B. Reduction of Yang-Mills fields

Here we outline the dimensional reduction of the Yang-Mills gauge fields. We recall the ten-dimensional action in eq. (5.4)

$$S_{CYM} = -\frac{1}{4} \int d^{10} x \sqrt{-\mathcal{G}} e^{-\Phi} \mathcal{F}^I{}_{\mu\nu} \mathcal{F}^{I\mu\nu} . \tag{B.1}$$

To proceed, we need to determine the Ansätze for the reduction of the vector fields $\mathcal{A}^I$. As we discussed in section 3, eqs. (2.3) and (3.1) comprise the ansatz for the case of flat internal torus. The ensuing mass terms m^I_{MN} already affect the gauge algebra, which becomes manifest when one considers the effect of Kaluza-Klein gauge transformations on the Yang-Mills fields, and the axions which arise from reducing these fields. For the scalar axion field, we had from (3.20)

$$\mathcal{A}'^I{}_M = \mathcal{A}^I{}_M + 2m^I_{MN} \omega^N . \tag{B.2}$$

The inhomogeneous term selects $\mathcal{A}^I{}_M$ as one of the axions of the theory. This must remain true even when the internal space is not flat, after appropriate generalizations of the gauge transformations are made, to account for γ^M_{NP}. This leads to the following ansatz for the Yang-Mills vector fields:

$$\mathcal{A}^I = \mathcal{A}^I{}_\mu(x)dx^\mu + \mathcal{A}^I{}_M(x)\eta^M + \sigma^I(y)\,, \tag{B.3}$$

where the forms σ^I are to be defined below. Let us discuss this ansatz. First, it ensures the correct transformation rules for $\mathcal{A}^I{}_M(x)$ after the reduction, so that it retains its role as the axion of the reduced theory. An alternative possibility, to which one might be tempted to resort, would have been to reduce the gauge field $\mathcal{A}^I$ such that the cross-term were $\bar{\mathcal{A}}^I{}_M\zeta^M$, where ζ^M were the invariant one-forms defined in (4.14). However, by the invariance of ζ^M under Kaluza-Klein gauge transformations, this would have implied that the reduced quantity $\bar{\mathcal{A}}^I{}_M$ would have been gauge singlets, and hence would not have transformed as given in (B.2) in the limit $\gamma^M_{NP} = 0$. On the other hand, (B.3) correctly produces (B.2) when $\gamma^M_{NP} \to 0$. Thus, the only possibility for reduction which could reduce to the correct limit γ^M_{NP} as defined in (B.2) is (B.3).

The fields σ^I generalize the terms $m^I_{MN}y^N$ in (3.1) to the case of a general internal space. They encode the information about the gauge field fluxes on internal two-cycles. This means, that if we consider the components of the gauge field strengths $\mathcal{F}^I$ in the internal space, we should expect to find $\mathcal{F}^I{}_{MN} \sim m^I_{MN}$ in the basis spanned by η^M's. Note that these fields can only be defined locally, and not globally. This is because we see that the mass terms must emerge as $m^I_{MN}\eta^M \wedge \eta^N \sim d\sigma^I$. Further, the mass terms must be harmonic forms by the projections of the gauge field equations of motion in the internal space: $d\mathcal{F}^I = 0$, $d\,{}^*\mathcal{F}^I \sim 0$ (where for simplicity we ignore the dilaton-dependent terms in the latter equation). Clearly, if mass terms were exact forms, as suggested by the derivation from $\sim d\sigma^I$, they would solve the equations of motion, but would have vanishing internal fluxes: $\int_{2-cycle}\mathcal{F}^I = 0$. Hence σ^I must be defined only locally, with nontrivial transition functions between different charts on the internal space. This of course merely generalizes the considerations involved in the construction of the Dirac monopole. However, this implies that the field $\mathcal{A}^I$ is also defined only locally, but in such a way that the perturbations around some background value are globally well-defined. Since the reduced fields are nothing else but such perturbations, they are well-defined in the reduced theory.

To determine the explicit form of σ^I when $\gamma^M_{NP} \neq 0$, we need to require that the gauge field strength evaluated from (B.3) has nonzero internal flux contributions. Since $\mathcal{F}^I = d\mathcal{A}^I$, we find

$$\mathcal{F}^I = \partial_{[\mu}\mathcal{A}^I{}_{\nu]}dx^\mu \wedge dx^\nu + \partial_\mu\mathcal{A}^I{}_M dx^\mu \wedge \eta^M - \mathcal{A}^I{}_M\gamma^M_{NP}\eta^N \wedge \eta^P + d\sigma^I\,. \tag{B.4}$$

This must coincide with

$$\mathcal{F}^I = \partial_{[\mu}\mathcal{A}^I{}_{\nu]}dx^\mu \wedge dx^\nu + \partial_\mu\mathcal{A}^I{}_M dx^\mu \wedge \eta^M - (m^I_{BC} + \mathcal{A}^I{}_M\gamma^M_{NP})\eta^N \wedge \eta^P\,, \tag{B.5}$$

leading to

$$d\sigma^I = -m^I_{MN}\eta^M \wedge \eta^N \tag{B.6}$$

This coincides with (3.4) in the limit $\gamma^M_{NP} = 0$. The quantities m^I_{NP} are constants which are antisymmetric in the lower two indices, and in the limit $\gamma^M_{NP} = 0$ they become identical with the axionic Yang-Mills masses discussed in Sec. (3). The integrability condition for (B.6), which is found by looking at

$$0 = d^2\sigma^I = 2m^I_{MN}\gamma^N_{PQ}\eta^M \wedge \eta^Q \wedge \eta^P\,, \tag{B.7}$$

where we have used (4.5), is more important, and is completely general. It comes from demanding that the field strength is independent of y^N. This leads to

$$m^I_{MN}\gamma^N_{PQ} + m^I_{PN}\gamma^N_{QM} + m^I_{QN}\gamma^N_{MP} = 0\,, \tag{B.8}$$

which will be interpreted as nothing else but one of the Jacobi identities for the structure constants of the reduced theory. Hence the information about the reduced nonabelian gauge symmetry is in fact encoded in the reduction ansatz, as a consistency condition between the various modes that are excited on the internal space. Comparing (B.3) with the ansatz for $\hat{\mathcal{A}}^I(x,y)$ in (3.1) shows that they coincide when $\gamma^M_{NP} = 0$, while the constraint (B.8) disappears. Hence indeed (B.3) and the solution of (B.6) together comprise the reduction ansatz for the Yang-Mills gauge fields.

With this ansatz, we can now reduce the action (B.1). We can reexpress the field strength (B.5) in terms of the tangent space basis (A.3) as

$$\begin{aligned}\mathcal{F}^I &= \frac{1}{2}\left(F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu}\right) dx^\mu \wedge dx^\nu + \mathcal{D}_\mu \mathcal{A}^I{}_M \mathcal{E}_A{}^M dx^\mu \wedge \mathcal{E}^A - \\ &\quad - \left(m^I_{MN} + \mathcal{A}^I{}_P\gamma^P_{MN}\right) \mathcal{E}_A{}^M \mathcal{E}_B{}^N \mathcal{E}^A \wedge \mathcal{E}^B\,,\end{aligned} \tag{B.9}$$

where we use

$$\begin{aligned}\mathcal{D}_\mu \mathcal{A}^I{}_M &= \partial_\mu \mathcal{A}^I{}_M - 2\left(m^I_{MN} + \mathcal{A}^I{}_P\gamma^P_{MN}\right) V^N{}_\mu \\ F^I{}_{\mu\nu} &= \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu - 2m^I_{MN} V^M{}_\mu V^N{}_\nu\end{aligned} \tag{B.10}$$

as well as the definition of the nonabelian Kaluza-Klein gauge field $V^M{}_{\mu\nu}$ given in (5.3). The reduced gauge fields $A^I{}_\mu$ are defined by

$$A^I{}_\mu = \mathcal{A}^I{}_\mu - \mathcal{A}^I{}_M V^M{}_\mu\,, \tag{B.11}$$

precisely as in (2.5), which diagonalizes the gauge field sector of the reduced theory. It then takes a very simple calculation to show that (B.9) gives the following expression for the reduced Yang-Mills action:

$$\begin{aligned}S_{CYM} = -\int d^Dx\sqrt{-g}e^{-\phi}\Big\{&\frac{1}{4}\left(F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu}\right)\left(F^{I\,\mu\nu} + \mathcal{A}^I{}_M V^{M\mu\nu}\right) + \\ &+ \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu \mathcal{A}^I{}_M \mathcal{D}^\mu \mathcal{A}^I{}_N + \\ &+ \mathcal{G}^{MP}\mathcal{G}^{NQ}\left(m^I_{MN} + \mathcal{A}^I{}_R\gamma^R_{MN}\right)\left(m^I_{PQ} + \mathcal{A}^I{}_S\gamma^S_{PQ}\right)\Big\}\,.\end{aligned} \tag{B.12}$$

That completes this step of the reduction procedure.

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C. Reduction of two-form

The reduction of the Kalb-Ramond two-form $\mathcal{B}$ contributions is done in a fashion similar to the reduction of the Yang-Mills gauge fields, except for the complications that are caused by the more involved tensor structure of $\mathcal{H}$ field, and the anomalous gauge transformation properties of $\mathcal{B}$, which require the inclusion of the Yang-Mills Chern-Simons terms in the definition of $\mathcal{H}$. The Kalb-Ramond action is

$$S_{NS} = -\frac{1}{12}\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}\,. \tag{C.1}$$

We can always write $\mathcal{B}$ in the form given in (2.3), but need to determine the proper ansatz for the y^M dependence such that the components of $\mathcal{B}$ produce internal fluxes of the three-form field strength. Further, we must ensure that when the limit $\gamma^M_{NP} = 0$ is taken, the ansatz correctly produces the gauge transformation properties given in (3.19) and (3.20). We can start in much the same way as we did previously, when we considered the Yang-Mills fluxes. First, we define the two-form potential

$$\hat{\mathcal{B}} = \mathcal{B} + \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma\,, \tag{C.2}$$

where the two-form $\mathcal{B}$ is independent of y^M, and hence can be viewed simply as a globally-defined perturbation around the Dirac-type configuration of $\hat{\mathcal{B}}$ producing an internal flux. This is encoded in the forms α, β and γ, which all depend on the internal coordinates. To define these forms, we recall that there are Yang-Mills Chern-Simons terms in the definition of the field strength $\mathcal{H}$: $\mathcal{H} = d\hat{\mathcal{B}} - \frac{1}{2}\mathcal{A}^I\wedge\mathcal{F}^I$ (2.2). In terms of the potential and field strength for the Yang-Mills fields (B.3) and (B.5) from the previous section, we find

$$\begin{aligned}
\mathcal{A}^I\wedge\mathcal{F}^I = {} & \frac{1}{2}\mathcal{A}^I{}_\mu\mathcal{F}^I{}_{\nu\lambda}dx^\mu\wedge dx^\nu\wedge dx^\lambda + \\
& + \left\{\frac{1}{2}\mathcal{F}^I{}_{\mu\nu}(\mathcal{A}^I{}_M+\sigma^I{}_M) + \mathcal{A}^I{}_\mu\partial_\nu\mathcal{A}^I{}_M\right\}dx^\mu\wedge dx^\nu\wedge\eta^M + \\
& + \left\{\partial_\mu\mathcal{A}^I{}_M\left(\mathcal{A}^I{}_N+\sigma^I{}_N\right) - \mathcal{A}^I{}_\mu\left(m^I_{MN}+\mathcal{A}^I{}_P\gamma^P_{MN}\right)\right\}dx^\mu\wedge\eta^M\wedge\eta^N - \\
& - \left(\mathcal{A}^I{}_M+\sigma^I{}_M\right)\left(m^I_{NP}+\mathcal{A}^I{}_Q\gamma^Q_{NP}\right)\eta^M\wedge\eta^N\wedge\eta^P\,.
\end{aligned} \tag{C.3}$$

In this expression, there is several terms which depend explicitly on the internal space coordinates y^M. However to be able to dimensionally reduce the theory, we must require that the three-form field strength must be y^M-independent. Hence the y^M-dependent terms in $d\mathcal{B}$ and $\mathcal{A}\mathcal{F}$ terms must cancel against each other, leaving only y^M-independent contributions in the components of $\mathcal{H}$ expressed in the covariant basis η^M (or the tangent basis (A.3), which is the equivalent and perhaps more usual statement).

For this purpose, we define the forms

$$\alpha = \mathcal{A}^I{}_M \eta^M \wedge \sigma^I , \qquad \beta = \mathcal{A}^I{}_\mu dx^\mu \sigma^I . \tag{C.4}$$

With this, we can after some algebra write down the expression for the three-form field strength $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H} = d\mathcal{B} &- \frac{1}{4}\mathcal{A}^I{}_\mu \mathcal{F}^I{}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\lambda - \\ &-\frac{1}{2}\left\{\frac{1}{2}\mathcal{F}^I{}_{\mu\nu}\mathcal{A}^I{}_M + \mathcal{A}^I{}_\mu \partial_\nu \mathcal{A}^I{}_M\right\} dx^\mu \wedge dx^\nu \wedge \eta^M + \\ &+\frac{1}{2}\left\{\partial_\mu \mathcal{A}^I{}_M \mathcal{A}^I{}_N - \mathcal{A}^I{}_\mu\left(2m^I_{MN} + \mathcal{A}^I{}_P \gamma^P_{MN}\right)\right\} dx^\mu \wedge \eta^M \wedge \eta^N + \\ &+\frac{1}{2}\mathcal{A}^I{}_M\left(2m^I_{NP} + \mathcal{A}^I{}_Q \gamma^Q_{NP}\right)\eta^M \wedge \eta^N \wedge \eta^P + \\ &+ d\gamma + \frac{1}{2}\sigma^I{}_M m^I_{NP} \eta^M \wedge \eta^N \wedge \eta^P \end{aligned} \tag{C.5}$$

and in this equation, only the last two terms contain explicit y^M dependence. This dependence however is now quite nontrivial. Namely, we can consider the exterior derivative of these two terms. We find

$$\begin{aligned} d^2\gamma + \frac{1}{2}d\left\{\sigma^I{}_M m^I_{NP}\eta^M \wedge \eta^N \wedge \eta^P\right\} &= -\frac{1}{2}d\sigma^I \wedge d\sigma^I \\ &= -\frac{1}{2}m^I_{[MN}m^I_{PQ]}\eta^M \wedge \eta^N \wedge \eta^P \wedge \eta^Q . \end{aligned} \tag{C.6}$$

On the other hand, the three-form field strength $\mathcal{H}$ must have components in the internal space which are independent of the internal space coordinates. Given the expression (C.5), that means that we must have

$$d\gamma + \frac{1}{2}\sigma^I{}_M m^I_{NP}\eta^M \wedge \eta^N \wedge \eta^P = \frac{1}{2}\beta_{MNP}\eta^M \wedge \eta^N \wedge \eta^P , \tag{C.7}$$

where β_{MNP} are constants as a consequence of equations of motion. Combining (C.6) and (C.7), we get the following constraint on the β_{MNP}: since $d\beta_{MNP}(\eta^M \wedge \eta^N \wedge \eta^P) = 3\beta_{MNP}\gamma^M_{QR}\eta^Q \wedge \eta^R \wedge \eta^N \wedge \eta^P$, comparing the sides of the equation we get

$$3\beta_{R[MN}\gamma^R_{PQ]} = m^I_{[MN}m^I_{PQ]} . \tag{C.8}$$

This is the last of the independent classes of Jacobi identities of the reduced theory.

Hence regardless of the specifics of the ansatz, that depends on the symmetry of the internal manifold, the general expression for $\mathcal{H}$ is inevitably

$$\begin{aligned}\mathcal{H} = &\left(\frac{1}{2}\partial_{[\mu}\mathcal{B}_{\nu\lambda]} - \frac{1}{4}\mathcal{A}^I{}_\mu \mathcal{F}^I{}_{\nu\lambda}\right) dx^\mu \wedge dx^\nu \wedge dx^\lambda + \\ &+\left(\partial_{[\mu}\mathcal{B}_{\nu]M} - \frac{1}{2}\left(\mathcal{F}^I{}_{\mu\nu}\mathcal{A}^I{}_M + \mathcal{A}^I{}_\mu \partial_\nu \mathcal{A}^I{}_M\right)\right) dx^\mu \wedge dx^\nu \wedge \eta^M + \\ &+\frac{1}{2}\left(\partial_\mu \mathcal{B}_{MN} - \mathcal{B}_{\mu P}\gamma^P_{MN} + \frac{1}{2}\left(\mathcal{A}^I{}_M \partial_\mu \mathcal{A}^I{}_N + \mathcal{A}^I{}_\mu (2m^I_{MN} + \mathcal{A}^I{}_P \gamma^P_{MN})\right)\right) dx^\mu \wedge \eta^M \wedge \eta^N + \\ &+\frac{1}{2}\mathcal{A}^I{}_M\left(2m^I_{NP} + \mathcal{A}^I{}_Q \gamma^Q_{NP}\right)\eta^M \wedge \eta^N \wedge \eta^P + \frac{1}{2}\beta_{MNP}\eta^M \wedge \eta^N \wedge \eta^P . \end{aligned} \tag{C.9}$$

Then to complete the reduction of the action (C.1), we need to rewrite (C.9) in the orthogonal basis (A.3). Using these definitions, we find after another lengthy calculation that

$$\begin{aligned}\mathcal{H} = &\frac{1}{6}H_{\mu\nu\lambda}dx^\mu\wedge dx^\nu\wedge dx^\lambda + \\ &+\frac{1}{2}\Big(H_{\mu\nu M} - \mathcal{A}^I_M F^I{}_{\mu\nu} - \mathcal{C}_{MN}V^N{}_{\mu\nu}\Big)\mathcal{E}_A{}^M dx^\mu\wedge dx^\nu\wedge\mathcal{E}^A + \\ &+\frac{1}{2}\Big(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\mathcal{E}_A{}^M\mathcal{E}_B{}^N dx^\mu\wedge\mathcal{E}^A\wedge\mathcal{E}^B + \\ &+\frac{1}{2}\Big(\beta_{MNP} + 2\mathcal{A}^I{}_M m^I_{NP} + 2\mathcal{C}_{MQ}\gamma^Q_{NP}\Big)\mathcal{E}_A{}^M\mathcal{E}_B{}^N\mathcal{E}_C{}^P\mathcal{E}^A\wedge\mathcal{E}^B\wedge\mathcal{E}^C\,. \qquad \text{(C.10)}\end{aligned}$$

Here we are using the new definitions

$$\begin{aligned}\mathcal{D}_\mu\mathcal{B}_{MN} &= \partial_\mu\mathcal{B}_{MN} + 2m^I_{MN}A^I{}_\mu + 2\gamma^P_{MN}B_{\mu P} - \\ &\quad -\beta_{MNP}V^P{}_\mu + 4\mathcal{B}_{Q[M}\gamma^Q_{N]P}V^P{}_\mu - 2\mathcal{A}^I{}_{[M}m^I_{N]P}V^P{}_\mu \\ H_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP}V^N{}_\mu V^P{}_\nu + \\ &\quad +4\gamma^P_{MN}B_{[\mu P}V^N{}_{\nu]} + 4m^I_{MN}A^I{}_{[\mu}V^N{}_{\nu]} \qquad \text{(C.11)}\end{aligned}$$

and

$$\begin{aligned}H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2}A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{2}V^M{}_\mu H_{\nu\lambda M} - \frac{1}{2}B_{\mu M}V^M{}_{\nu\lambda} + \frac{1}{2}\beta_{MNP}V^M{}_\mu V^N{}_\nu V^P{}_\lambda - \\ -m^I_{MN}A^I{}_\mu V^M{}_\nu V^N{}_\lambda - \gamma^M_{NP}B_{\mu M}V^N{}_\nu V^P{}_\lambda + \text{ cyclic perm.} \qquad \text{(C.12)}\end{aligned}$$

in addition to (4.15), (4.22), and (2.5). We have redefined $B_{\mu M}$ and $B_{\mu\nu}$ according to (2.5), in order to express the reduced action, in a manifestly gauge- and duality-symmetric way. With this, we finally find the reduced Kalb-Ramond action in D dimensions:

$$\begin{aligned}S_{KR} = -\int d^Dx\sqrt{-g}e^{-\phi}\Bigg\{&\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \\ &+\frac{1}{4}\mathcal{G}^{MN}\left(B_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MP}V^P{}_{\mu\nu}\right)\left(B^{\mu\nu}{}_N - \mathcal{A}^I{}_N F^{I\;\mu\nu} - \mathcal{C}_{NQ}V^{Q\mu\nu}\right) + \\ &+\frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\left(\mathcal{D}_\mu B_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\right)\left(\mathcal{D}^\mu B_{PQ} + \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\right) + \\ &+\frac{3}{4}\mathcal{G}^{MQ}\mathcal{G}^{NR}\mathcal{G}^{PS}\left(\beta_{MNP} + 2\mathcal{A}^I{}_{[M}m^I_{NP]} - 2\mathcal{C}_{T[M}\gamma^G_{NP]}\right)\times \\ &\qquad\times\left(\beta_{QRS} + 2\mathcal{A}^J{}_{[Q}m^J_{RS]} - 2\mathcal{C}_{U[Q}\gamma^H_{RS]}\right)\Bigg\}\,. \qquad \text{(C.13)}\end{aligned}$$

This is the last step in the reduction of the effective action.

D. Gauge algebra

Here we give a detailed derivation of the gauge algebra (5.17). Our approach is to first compute the gauge-dependent variations of the reduced gauge fields under the gauge transformations, and then to reassemble them into the algebra using the

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homomorphism between the neighbourhood of identity of the gauge group and the gauge algebra. We will work in the ascending order of complexity. Thus, we start with the reduced form of the gauge transformations associated with the Kalb-Ramond two-form field.

Recall that the two-form field introduces a one-form gauge symmetry in the original ten-dimensional theory, $\mathcal{B} \to \mathcal{B}' = \mathcal{B} + d\Lambda$, where $\Lambda = \lambda_M \eta^M + \lambda_\mu dx^\mu$ is a one-form. Hence, upon dimensional reduction to D dimensions, we find d new U(1) gauge symmetries, counted by the components of Λ in the internal space. Of course, there still remains the reduced one-form of the original gauge transformation $\Lambda = \lambda_\mu dx^\mu$, which produces variations of the reduced field $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$. These transformations must allow the reduced fields of the theory, which emerge from the two-form, to retain their y^M-independent form, in order that the reduction is consistent. Hence all such reduced gauge transformations must be independent of y^M as well. The physical meaning of this statement can be seen as follows: From the point of view of the reduced theory, the modes which are excited in the internal dimensions (i.e. they depend on y^M) are all very heavy, and hence cannot be excited by the low energy physics, which is described by the effective action in D dimensions. So from this and the reduction ansatz for the two-form, we see that the only fields which transform nontrivially under the gauge transformations generated by λ_M are the reduced components $\mathcal{B}_{\mu\nu}$, $\mathcal{B}_{\mu M}$ and $\mathcal{B}_{MN}$. An explicit computation shows that (where we set the reduced two-form gauge transformation to zero, $\lambda_\mu = 0$)

$$\begin{aligned} \mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda_P \gamma^P_{MN} \\ \mathcal{B}'_{\mu M} &= \mathcal{B}_{\mu M} + \partial_\mu \lambda_M \\ \mathcal{B}'_{\mu\nu} &= \mathcal{B}_{\mu\nu} \,. \end{aligned} \tag{D.1}$$

Using field redefinitions (2.5) which diagonalize the reduced gauge transformations, we can rewrite the previous equation as

$$\begin{aligned} \mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda_P \gamma^P_{MN} \\ B'_{\mu A} &= B_{\mu A} + \partial_\mu \lambda_M - 2\lambda_P \gamma^P_{MN} V^N{}_\mu \\ B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda_M V^M{}_{\mu\nu} + \gamma^M_{NP} \lambda_M V^N{}_\mu V^P{}_\nu \,. \end{aligned} \tag{D.2}$$

It is clear that these are the only nontrivial gauge transformation rules in the case of gauge symmetries arising from the two-form sector. Note that the $B_{\mu\nu}$ transforms nontrivially, which is a signature of the anomaly that emerges from the reduction.

Next, we turn to the reduction of the Yang-Mills subalgebra of the original ten-dimensional gauge group. In the ten-dimensional case, these symmetries correspond to $\mathcal{A}^I \to \mathcal{A}'^I = \mathcal{A}^I + d\Lambda^I$. If we wish to preserve the reduction ansatz, and the fact that the reduced field strengths are independent of y^M, again we must require that the residual gauge symmetries after dimensional reduction are independent of y^M. In the Yang-Mills sector after the reduction, and diagonalization of the reduced degrees of freedom using (2.5), it is easy to see that

$$\mathcal{A}'^I{}_M = \mathcal{A}^I{}_M\,, \qquad \mathcal{A}'^I{}_\mu = \mathcal{A}^I{}_\mu + \partial_\mu \lambda^I\,. \tag{D.3}$$

These transformations obviously leave the Kaluza-Klein sector unchanged. However, they induce nontrivial transformation properties in the two-form sector. To start with, due to the anomaly, the two-form field in ten-dimensions transforms according to $\hat{\mathcal{B}}' = \hat{\mathcal{B}} + \frac{1}{2}\lambda^I \mathcal{F}^I + d\Lambda$. If we consider the decomposition of $\hat{\mathcal{B}}$ in terms of the reduced degrees of freedom, we see that in addition to the $\mathcal{B}$ terms, also the terms proportional to $\mathcal{A}^I{}_\mu$ are gauge-noninvariant. Hence we must have the following transformation rule for the combination of these fields:

$$\begin{aligned}
&\frac{1}{2}\mathcal{B}'_{\mu\nu}(x)dx^\mu \wedge dx^\nu + \mathcal{B}'_{\mu M}(x)dx^\mu \wedge \eta^M + \frac{1}{2}\mathcal{B}'_{MN}(x)\eta^M \wedge \eta^N + \mathcal{A}'^I{}_\mu dx^\mu \sigma^I = \\
&\qquad = \frac{1}{2}\mathcal{B}_{\mu\nu}(x)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu M}(x)dx^\mu \wedge \eta^M + \frac{1}{2}\mathcal{B}_{MN}(x)\eta^M \wedge \eta^N \\
&\qquad\quad + \mathcal{A}'^I{}_\mu dx^\mu \sigma^I + \frac{1}{2}\lambda^I \mathcal{F}^I + d\Lambda\,,
\end{aligned} \tag{D.4}$$

Next, using the fact that

$$\mathcal{A}'^I{}_\mu dx^\mu \wedge \sigma^I = \mathcal{A}^I{}_\mu dx^\mu \wedge \sigma^I + d(\lambda^I \sigma^I) + \lambda^I m^I_{MN}\eta^M \wedge \eta^N\,, \tag{D.5}$$

where we have integrated the gauge-dependent piece by parts, and have used $d\sigma^I = -m^I_{MN}\eta^M \wedge \eta^N$, we get that

$$\begin{aligned}
&\frac{1}{2}\mathcal{B}'_{\mu\nu}(x)dx^\mu \wedge dx^\nu + \mathcal{B}'_{\mu M}(x)dx^\mu \wedge \eta^M + \frac{1}{2}\mathcal{B}'_{MN}(x)\eta^M \wedge \eta^N = \\
&\qquad = \frac{1}{2}\mathcal{B}_{\mu\nu}(x)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu M}(x)dx^\mu \wedge \eta^M + \frac{1}{2}\mathcal{B}_{MN}(x)\eta^M \wedge \eta^N + \\
&\qquad\quad + \frac{1}{2}\lambda^I(\mathcal{F}^I - m^I_{MN}\eta^M \wedge \eta^N) + d\left(\Lambda - \frac{1}{2}\lambda^I \sigma^I\right) \\
&\qquad = \frac{1}{2}\mathcal{B}_{\mu\nu}(x)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu M}(x)dx^\mu \wedge \eta^M + \frac{1}{2}\mathcal{B}_{MN}(x)\eta^M \wedge \eta^N + \\
&\qquad\quad + \frac{1}{4}\lambda^I \mathcal{F}^I{}_{\mu\nu}dx^\mu \wedge dx^\nu + \frac{1}{2}\lambda^I \partial \mathcal{A}^I{}_M dx^\mu \wedge \eta^M - \\
&\qquad\quad - \frac{1}{2}\lambda^I\left(2m^I_{MN} + \mathcal{A}^I{}_P \gamma^P_{MN}\right)\eta^M \wedge \eta^N + d\left(\Lambda - \frac{1}{2}\lambda^I \sigma^I\right).
\end{aligned} \tag{D.6}$$

Now, using this equation and the formulae for diagonalizing the reduced degrees of freedom given in (2.5), after some tedious but straightforward algebra we finally arrive at

$$\begin{aligned}
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda^I m^I_{MN} \\
A'^I{}_\mu &= A^I{}_\mu + \partial_\mu \lambda^I \\
B'_{\mu M} &= B_{\mu M} - 2\lambda^I m^I_{MN} V^N{}_\mu \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda^I F^I{}_{\mu\nu} + m^I_{MN}\lambda^I V^M{}_\mu V^N{}_\nu\,,
\end{aligned} \tag{D.7}$$

which as we see are independent of γ^M_{NP}, and hence are identical to the simple case of a flat internal torus.

Finally, we consider the reduction of the Kaluza-Klein gauge transformations. They have the most complicated structure, since they affect the matter degrees of freedom from all three sectors. We start with the metric moduli $\mathcal{G}_{MN}$ and the gauge fields $V^M{}_\mu$. In fact, we have already determined how these fields transform; the results are given in (4.16) and (4.17)

$$\mathcal{G}'{}_{MN} = \mathcal{S}_M{}^P \mathcal{S}_N{}^Q \mathcal{G}_{PQ}\,, \qquad V'^M{}_\mu = \mathcal{S}^M{}_N V^N{}_\mu + \mathcal{O}^M{}_N \partial_\mu \omega^N\,. \tag{D.8}$$

Using the infinitesimal forms of $\mathcal{O}^M{}_N$ and $\mathcal{S}^M{}_N$ given in (4.10), it is straightforward to compute the infinitesimal form of the gauge transformations of $\mathcal{G}_{MN}$ and $V^M{}_\mu$. They are

$$\begin{aligned} \mathcal{G}'{}_{MN} &= \mathcal{G}_{MN} + 2\gamma^P_{MQ}\omega^Q \mathcal{G}_{PN} + 2\gamma^P_{NQ}\omega^Q \mathcal{G}_{MP} + \mathrm{O}(\omega^2) \\ V'^M{}_\mu &= V^M{}_\mu - 2\gamma^M_{NP}\omega^P V^N{}_\mu + \partial_\mu \omega^M + \mathrm{O}(\omega^2)\,. \end{aligned} \tag{D.9}$$

From the transformation rule for the moduli matrix $\mathcal{G}_{MN}$, we see that it is a direct product of scalars belonging to the singlet and two adjoints (with charge -1) of the isometry group. This is precisely the transformation rule of the axions hidden in $\mathcal{G}_{MN}$. They can be explicitly found by Gauss decomposition of the matrix $\mathcal{E}^A{}_M$ (with the property that $\mathcal{G}_{MN} = \delta_{AB}\mathcal{E}^A{}_M \mathcal{E}^B{}_N$) and identifying the pivots with the dilaton-like fields and the upper triangular matrix elements with the axions — see, e.g., eq. (4.20).

Next we consider the effect of Kaluza-Klein gauge transformations on Yang-Mills gauge fields. From (4.22), it is clear that the reduced fields transform nontrivially under the Kaluza-Klein transformations. A lengthy but straightforward calculation shows that due to the form of the compensator forms σ^I, we find

$$\sigma'^I = \sigma^I + 2m^I_{MN}\omega^M \eta^N + d\,\Xi\,. \tag{D.10}$$

Hence if we accompany the Kaluza-Klein gauge transformation by a Yang-Mills custodian transformation

$$\mathcal{A}'^I = \mathcal{A}^I + d\,\Xi \tag{D.11}$$

the components of $\mathcal{A}$ transform infinitesimally according to

$$\begin{aligned} \mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2m^I_{MN}\omega^N + 2\gamma^N_{MP}\mathcal{A}^I{}_N \omega^P + \mathrm{O}(\omega^2) \\ \mathcal{A}'^I{}_\mu &= \mathcal{A}^I{}_\mu + \mathcal{A}^I{}_M \partial_\mu \omega^M + \mathrm{O}(\omega^2)\,, \end{aligned} \tag{D.12}$$

so that the diagonalized reduced Yang-Mills degrees of freedom transform according to

$$\begin{aligned} \mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2\gamma^N_{MP}\omega^P \mathcal{A}^I{}_N + 2m^I_{MN}\omega^N + \mathrm{O}(\omega^2) \\ A'^I{}_\mu &= A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + \mathrm{O}(\omega^2)\,. \end{aligned} \tag{D.13}$$

Finally we come to consider the transformation properties of the two-form degrees of freedom. First, we recall the notation of (C.2), $\hat{\mathcal{B}} = \mathcal{B} + \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma$, and the definition (C.4). Note that $\alpha + \beta = \mathcal{A}^I \wedge \sigma^I$. Now we use an algebraic trick to compute the gauge dependence of the compensator forms γ. Instead of working with γ itself, we consider its exterior derivative. Using the fact that

$$d\gamma = -\frac{1}{2} m^I_{MN} \sigma^I \wedge \eta^M \wedge \eta^N + \frac{1}{2}\beta_{MNP}\eta^M \wedge \eta^N \wedge \eta^P \tag{D.14}$$

if we denote the gauge variation $\delta\gamma = \gamma' - \gamma$ we find after a straightforward calculation

$$d\delta\gamma = m^I_{MN}\sigma^I \wedge d(\omega^M \eta^N) + \frac{1}{2}\delta\sigma^I \wedge \sigma^I + \frac{1}{2}\beta_{MNP}\delta(\eta^M \wedge \eta^N \wedge \eta^P)\,. \tag{D.15}$$

Next, using the transformation properties of σ^I, given in (D.10), we can deduce that the explicit form of the exterior derivative of the gauge dependence of γ is

$$\begin{aligned} d\delta\gamma = & -d\vartheta + 2m^I_{MN} m^I_{PQ} \omega^N \eta^M \wedge \eta^P \wedge \eta^Q - \\ & -3\beta_{MNP}\gamma^M_{QR}\omega^R \eta^Q \wedge \eta^N \wedge \eta^P - \frac{3}{2}\beta_{MNP} d\omega^M \wedge \eta^N \wedge \eta^P\,, \end{aligned} \tag{D.16}$$

where

$$\vartheta = \sigma^I \wedge \left(m^I_{MN}\omega^M \eta^N - \frac{1}{2} d\,\Xi\right). \tag{D.17}$$

Now we can integrate eq. (D.16). Using the Jacobi identity (C.8), we find after some straightforward algebra that

$$\gamma' = \gamma - \vartheta - \frac{3}{2}\beta_{MNP}\omega^M \eta^N \wedge \eta^P + d\tilde{\Lambda}\,, \tag{D.18}$$

where $\tilde{\Lambda}$ is an arbitrary globally defined one-form. Then, using (D.10) and (D.12), we can compute the gauge variation of $\alpha + \beta$. We find

$$\begin{aligned} \alpha' + \beta' = & \ \alpha + \beta + 2m^I_{MN}\omega^M \sigma^I \wedge \eta^N + \\ & + \left(\mathcal{A}^I{}_\mu dx^\mu + \mathcal{A}^I{}_M \eta^M\right) \wedge \left(2m^I_{MN}\omega^M \eta^N + d\,\Xi\right). \end{aligned} \tag{D.19}$$

We can now compute the transformation properties of the reduced components. Recalling that the Yang-Mills fields should be gauge transformed by the custodian transformation (D.11), which induces $\delta\mathcal{B} = \frac{1}{2}\lambda^I \mathcal{F}^I$, after some more algebra we find that

$$\begin{aligned} \mathcal{B}' = & \ \mathcal{B} + \delta\mathcal{B} - \delta\gamma - \frac{1}{2}\delta(\alpha + \beta) \\ = & \ d\Big(\Lambda - \tilde{\Lambda} + \frac{1}{2}\mathcal{A}^I \wedge \Xi\Big) + \\ & + \frac{3}{2}\beta_{MNP}\omega^M \eta^N \wedge \eta^P - \left(\mathcal{A}^I{}_\mu dx^\mu + \mathcal{A}^I{}_M \eta^M\right) m^I_{NP}\omega^N \eta^P\,. \end{aligned} \tag{D.20}$$

We should mention again here that the role of the custodian transformations, as well as the anomaly, is essential to ensure the cancellation of the y^M-dependent terms in the gauge transformation laws. This is precisely what will eventually produce the nonabelian symmetry structure among the reduced vector fields. Choosing the $\mathcal{B}$ field gauge transformation according to $\Lambda = \tilde{\Lambda} - \frac{1}{2}\mathcal{A}^I \wedge \Xi$, we finally find that the infinitesimal transformations of $\mathcal{B}$ are

$$\mathcal{B}' = \mathcal{B} + \frac{3}{2}\beta_{MNP}\omega^M \eta^N \wedge \eta^P - \left(\mathcal{A}^I{}_\mu dx^\mu + \mathcal{A}^I{}_M \eta^M\right) m^I_{NP}\omega^N \eta^P + \mathrm{O}(\omega^2)\,. \quad \text{(D.21)}$$

Using the expressions for the diagonalized reduced degrees of freedom (2.5), after a straightforward calculation we finally find

$$\begin{aligned}
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} + 3\beta_{MNP}\omega^P + 2a^I{}_{[M} m^I_{N]P}\omega^P + \mathrm{O}(\omega^2) \\
B'_{\mu M} &= B_{\mu M} + 2\gamma^N_{MP}\omega^P B_{\mu N} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + \mathrm{O}(\omega^2) \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu - \\
&\quad -2\gamma^P_{MN}\omega^M B_{[\mu|P} V^N{}_{\nu]} - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + \mathrm{O}(\omega^2)
\end{aligned} \quad \text{(D.22)}$$

So the nontrivial infinitesimal Kaluza-Klein transformations for all the reduced degrees of freedom are

$$\begin{aligned}
\mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2\gamma^N_{MP}\omega^P \mathcal{A}^I{}_N + 2m^I_{MN}\omega^N \\
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} + 3\beta_{MNP}\omega^P + 2a^I{}_{[M} m^I_{N]P}\omega^P + \mathrm{O}(\omega^2) \\
\mathcal{G}'_{MN} &= \mathcal{G}_{MN} + 2\gamma^P_{MQ}\omega^Q \mathcal{G}_{PN} + 2\gamma^P_{NQ}\omega^Q \mathcal{G}_{MP} + \mathrm{O}(\omega^2) \\
V'^M{}_\mu &= V^M{}_\mu - 2\gamma^M_{NP}\omega^P V^N{}_\mu + \partial_\mu \omega^M + \mathrm{O}(\omega^2) \\
A'^I{}_\mu &= A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + \mathrm{O}(\omega^2) \\
B'_{\mu M} &= B_{\mu M} + 2\gamma^N_{MP}\omega^P B_{\mu N} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + \mathrm{O}(\omega^2) \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu \\
&\quad -2\gamma^P_{MN}\omega^M B_{[\mu|P} V^N{}_{\nu]} - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + \mathrm{O}(\omega^2)
\end{aligned} \quad \text{(D.23)}$$

We are now finally ready to reassemble the results of this appendix into the algebra of the reduced gauge theory. The calculation is fairly straightforward, and hence we merely describe the method here and list the results. First, we recapitulate the explicit form of the gauge group in the neighbourhood of identity. We have

1. reduced two-form gauge transformations:

$$\begin{aligned}
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda_P \gamma^P_{MN} \\
B'_{\mu M} &= B_{\mu M} + \partial_\mu \lambda_M - 2\lambda_P \gamma^P_{MN} V^N{}_\mu \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda_M V^M{}_{\mu\nu} + \gamma^M_{NP}\lambda_M V^N{}_\mu V^P{}_\nu
\end{aligned} \quad \text{(D.24)}$$

2. reduced Yang-Mills gauge transformations:

$$
\begin{aligned}
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} - 2\lambda^I m^I_{MN} \\
A'^I{}_\mu &= A^I{}_\mu + \partial_\mu \lambda^I \\
B'_{\mu M} &= B_{\mu M} - 2\lambda^I m^I_{MN} V^N{}_\mu \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\lambda^I F^I{}_{\mu\nu} + m^I_{MN}\lambda^I V^M{}_\mu V^N{}_\nu
\end{aligned}
\tag{D.25}
$$

3. reduced Kaluza-Klein gauge transformations:

$$
\begin{aligned}
\mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2\gamma^N_{MP}\omega^P \mathcal{A}^I{}_N + 2m^I_{MN}\omega^N \\
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} + 3\beta_{MNP}\omega^P + 2a^I{}_{[M} m^I_{N]P}\omega^P + \mathrm{O}(\omega^2) \\
\mathcal{G}'{}_{MN} &= \mathcal{G}_{MN} + 2\gamma^P_{MQ}\omega^Q \mathcal{G}_{PN} + 2\gamma^P_{NQ}\omega^Q \mathcal{G}_{MP} + \mathrm{O}(\omega^2) \\
V'^M{}_\mu &= V^M{}_\mu - 2\gamma^M_{NP}\omega^P V^N{}_\mu + \partial_\mu \omega^M + \mathrm{O}(\omega^2) \\
A'^I{}_\mu &= A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + \mathrm{O}(\omega^2) \\
B'_{\mu M} &= B_{\mu M} + 2\gamma^N_{MP}\omega^P B_{\mu N} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + \mathrm{O}(\omega^2) \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu - \\
&\quad -2\gamma^P_{MN}\omega^M B_{[\mu|P} V^N{}_{\nu]} - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + \mathrm{O}(\omega^2)
\end{aligned}
\tag{D.26}
$$

We can now calculate the algebra of gauge generators. As in section 3, the group generators are denoted T_a, where the indices take values in the space of $\{{}_M, {}^M, {}^I\}$, with dimension $2d{+}16$, and in the explicit form arise from the three sectors described above: $T_a = (Z_M, X^M, Y^I)$. The gauge algebra must close, and hence it satisfies

$$
[T_a, T_b] = i f_{ab}{}^c T_c \,, \tag{D.27}
$$

where $f_{ab}{}^c$ are the structure constants which we need to determine. We can compute them by considering the products of transformations (D.24)-(D.26), which are in general of the form $h^{-1}\cdot g^{-1}\cdot h\cdot g$, with h and g any two infinitesimal gauge transformations. Infinitesimally, for any two operators A, B and a number $\alpha \ll 1$, we have $e^{\alpha A} B e^{-\alpha A} = B + \alpha[A,B] + \mathrm{O}(\alpha^2)$. So using $g = \exp(i\hat{\omega}^a_1 T_a)$ and $h = \exp(i\hat{\omega}^a_2 T_a)$, where $\hat{\omega}^a = (\omega^M, \lambda_M, \lambda^I)$ are the gauge transformation parameters defined above, we have, to linear order in $\hat{\omega}^a_1$,

$$
g^{-1}\cdot h\cdot g = h - i\hat{\omega}^a_1 [T_a, h] \tag{D.28}
$$

and hence we find, to the lowest nontrivial order (quadratic)

$$
h^{-1}\cdot g^{-1}\cdot h\cdot g = 1 + \hat{\omega}^a_1\hat{\omega}^b_2 [T_a, T_b] = 1 + i f_{ab}{}^c \hat{\omega}^a_1 \hat{\omega}^b_2 T_c \,. \tag{D.29}
$$

Hence substituting the explicit form of the gauge transformations (D.24)–(D.26), we can calculate the structure constants. To do it, we must project the algebra composition laws on the set of basis states of our irreducible Lorentz representations, i.e. we will compute $h^{-1} \cdot g^{-1} \cdot h \cdot g|\Psi\rangle$. At first glance, it may be tempting to apply gauge transformations to scalar fields, because these are simpler. However, while the scalars transform nontrivially, in general they do not span a faithful representation of the gauge group. Thus we evaluate $h^{-1} \cdot g^{-1} \cdot h \cdot g|\Psi\rangle$ on the set of basis states defined by the vector fields, because they do span a faithful representation of the gauge group. Clearly, we should consider terms of the form $[X, X]$, $[Y, Y]$, $[Z, Z]$, $[X, Y]$, $[X, Z]$ and $[Y, Z]$. The explicit computation, which can be carried out straightforwardly using the formulas above, produces the structure constants. They are

$$\begin{aligned} f^{M}{}_{NP} &= f_{NP}{}^{M} = 2\gamma^{M}_{NP}\,, \\ f^{I}{}_{MN} &= f_{MN}{}^{I} = 2m^{I}_{MN}\,, \\ f_{MNP} &= -3\beta_{MNP} \end{aligned} \tag{D.30}$$

With this, we can finally write down the explicit form of the reduced gauge algebra:

$$\begin{aligned} [X^M, X^N] &= [Y^I, Y^J] = [X^M, Y^I] = 0 \\ [X^M, Z_N] &= 2i\gamma^{M}_{NP} X^P \\ [Y^I, Z_M] &= 2im^{I}_{MN} X^N \\ [Z_M, Z_N] &= -3i\beta_{MNP} X^P + 2im^{I}_{MN} Y^I + 2i\gamma^{P}_{MN} Z_P\,. \end{aligned} \tag{D.31}$$

E. Generalized axion reductions of ref. [6]

Above, we noted that the ansatz (3.1) is linear in the internal coordinates despite the explicit presence of the axionic potentials in the action, and not just their derivatives. Naively, this may seem to contradict our discussion of ref. [6]. There, we considered a similar situation, where the explicit appearance of undifferentiated axion potentials required that to generate all the axion masses one had to introduce an ansatz where the dependence on the internal space coordinates was of order higher than linear. We explicitly considered a case in which the reduction ansatz was quadratic in internal coordinates. However, we also pointed out that these quadratic terms could be removed by a suitable field redefinition. To clarify the situation, let us re-examine that case here. There were three scalar axions $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ appearing as off-diagonal metric components[4] in a reduction on T^3. The corresponding "field strengths," which naturally appeared in the dimensional reduction, were

$$\mathcal{F}_1 = d\mathcal{A}_1\,, \qquad \mathcal{F}_2 = d\mathcal{A}_2\,, \qquad \mathcal{F}_3 = d\mathcal{A}_3 - \mathcal{A}_2 d\mathcal{A}_1\,. \tag{E.1}$$

[4]The fields in ref. [6] and above in eq. (4.20) correspond to those here as follows: $\mathcal{A}_1 = \mathcal{A}_0^{(12)}$, $\mathcal{A}_2 = \mathcal{A}_0^{(23)}$ and $\mathcal{A}_3 = \mathcal{A}_0^{(13)}$.

As we have showed in [6], the ansatz which simultaneously induces all three possible mass terms is

$$\begin{aligned}\mathcal{A}_1(x,y) &= \mathcal{A}_1(x) + m_1 y\\ \mathcal{A}_2(x,y) &= \mathcal{A}_2(x) + m_2 y\\ \mathcal{A}_3(x,y) &= \mathcal{A}_3(x) + m_3 y + m_2 y \mathcal{A}_1(x) + \frac{1}{2} m_1 m_2 y^2\,.\end{aligned} \tag{E.2}$$

A direct substitution of (E.2) into (E.1) shows that the field strengths do not depend on the coordinate y.

However, it is easy to see that replacing the axion $\mathcal{A}_3$ by

$$\mathcal{A}_3 = \bar{\mathcal{A}}_3 + \frac{1}{2}\mathcal{A}_1\mathcal{A}_2 \tag{E.3}$$

redefines its field strength to

$$\mathcal{F}_3 = d\bar{\mathcal{A}}_3 + \frac{1}{2}\mathcal{A}_1 d\mathcal{A}_2 - \frac{1}{2}\mathcal{A}_2 d\mathcal{A}_1 \tag{E.4}$$

and also completely removes the terms of order of y^2 in the reduction ansatz. Indeed, the last formula in (E.2) is replaced by

$$\bar{\mathcal{A}}_3(x,y) = \mathcal{A}_3(x) + m_3 y + \frac{1}{2} m_2 y \mathcal{A}_1(x) - \frac{1}{2} m_1 y \mathcal{A}_2(x)\,. \tag{E.5}$$

Note that the (anti)symmetric form of the "Chern-Simons" contribution to $\mathcal{F}_3$ in (E.4) closely resembles that in the three-form field strength (2.15), in the case of interest in the present paper. This seems to be the essential ingredient in the linearization of the reduction ansatz. Note that the democratic form of the revised ansatz (E.5) is also similar to that for $\hat{\mathcal{B}}_{MN}$ in eq. (3.1).

The "meaning" of the disappearance of the $\mathrm{O}(y^2)$ terms can be deduced as follows: The quadratic ansatz (E.2) can be rewritten in the matrix form [6] as[5]

$$\begin{aligned}\begin{pmatrix} 1 & \mathcal{A}_1(x,y) & \mathcal{A}_3(x,y)\\ 0 & 1 & \mathcal{A}_2(x,y)\\ 0 & 0 & 1\end{pmatrix} &= \begin{pmatrix} 1 & \mathcal{A}_1(x) & \mathcal{A}_3(x)\\ 0 & 1 & \mathcal{A}_2(x)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & m_1 y & m_3 y + \frac{m_1 m_2}{2} y^2\\ 0 & 1 & m_2 y\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} 1 & \mathcal{A}_1(x) & \mathcal{A}_3(x)\\ 0 & 1 & \mathcal{A}_2(x)\\ 0 & 0 & 1\end{pmatrix} \exp\left[y\begin{pmatrix} 0 & m_1 & m_3\\ 0 & 0 & m_2\\ 0 & 0 & 0\end{pmatrix}\right].\end{aligned} \tag{E.6}$$

Noting that for $\mathcal{A}_3 = \bar{\mathcal{A}}_3 + \frac{1}{2}\mathcal{A}_1\mathcal{A}_2$, we can relate the field redefinitions in the matrix form as well

$$\exp\begin{pmatrix} 0 & \mathcal{A}_1 & \bar{\mathcal{A}}_3\\ 0 & 0 & \mathcal{A}_2\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix} 1 & \mathcal{A}_1 & \bar{\mathcal{A}}_3 + \frac{\mathcal{A}_1\mathcal{A}_2}{2}\\ 0 & 1 & \mathcal{A}_2\\ 0 & 0 & 1\end{pmatrix}, \tag{E.7}$$

[5]These matrices correspond to the dreibein of the internal torus [6], where we have set to zero the dilatonic fields that determine the scale of the internal dimensions. Alternatively, the latter can be factored out as in eq. (4.20). The fields of eq. (3.1) however all come from the p-forms of the theory. Nevertheless the reduction algorithms are closely related.

we can rewrite the reduction ansatz as

$$\exp\begin{pmatrix}0 & \mathcal{A}_1(x,y) & \bar{\mathcal{A}}_3(x,y)\\ 0 & 0 & \mathcal{A}_2(x,y)\\ 0 & 0 & 0\end{pmatrix} = \exp\begin{pmatrix}0 & \mathcal{A}_1(x) & \bar{\mathcal{A}}_3(x)\\ 0 & 0 & \mathcal{A}_2(x)\\ 0 & 0 & 0\end{pmatrix}\exp\left[y\begin{pmatrix}0 & m_1 & m_3\\ 0 & 0 & m_2\\ 0 & 0 & 0\end{pmatrix}\right]. \tag{E.8}$$

Now, we recall a version of the Baker-Campbell-Hausdorff formula, which allows us to evaluate products of exponentials of matrices. Namely, for as long as $[A,[A,B]] = [B,[A,B]] = 0$ for any two matrices A, B we have $\exp(A)\exp(yB) = \exp(A + yB + \frac{1}{2}y[A,B])$. In our case,

$$A = \begin{pmatrix}0 & \mathcal{A}_1(x) & \bar{\mathcal{A}}_3(x)\\ 0 & 0 & \mathcal{A}_2(x)\\ 0 & 0 & 0\end{pmatrix} \qquad B = \begin{pmatrix}0 & m_1 & m_3\\ 0 & 0 & m_2\\ 0 & 0 & 0\end{pmatrix}$$
$$[A,B] = \begin{pmatrix}0 & 0 & m_2\mathcal{A}_1(x) - m_1\mathcal{A}_2(x)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{E.9}$$

and hence using the Baker-Campbell-Hausdorff formula, upon the substitution into (E.8) and taking the logarithm of both sides, we find

$$\begin{pmatrix}0 & \mathcal{A}_1(x,y) & \bar{\mathcal{A}}_3(x,y)\\ 0 & 0 & \mathcal{A}_2(x,y)\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \mathcal{A}_1(x) + m_1 y & \bar{\mathcal{A}}_3(x) + \frac{1}{2}y(m_2\mathcal{A}_1(x) - m_1\mathcal{A}_2(x))\\ 0 & 0 & \mathcal{A}_2(x) + m_2 y\\ 0 & 0 & 0\end{pmatrix} \tag{E.10}$$

which is precisely the linear reduction ansatz. Hence, we see that the $\mathrm{O}(y^2)$ disappeared essentially by making the field redefinition which amounts to taking a functional logarithm of the axionic degrees of freedom. Nevertheless, we note that in general this procedure of linearization may turn out to be quite awkward.

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